3 (More on) The Stress Tensor and the Navier-Stokes Equations

3.1 The symmetry of the stress tensor

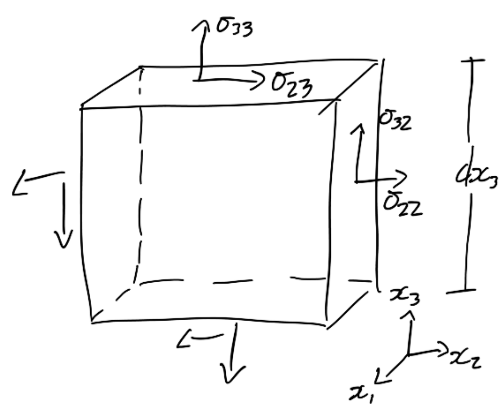
In principle, the stress tensor has nine independent components. BUT only six of these are independent. That is because the off-diagonal elements (those representing tangent or shear stresses as opposed to normal stresses) must satisfy a symmetry condition, $\sigma_{ij} = \sigma_{ji}$. That is,

$$\sigma_{12} = \sigma_{21}$$

$$\sigma_{23} = \sigma_{32}$$

$$\sigma_{13} = \sigma_{31}$$

An elementary derivation of this result comes by considering a cube of fluid and examining the torques around the centre of gravity.



Since the torque of the body forces go through the centre of gravity and the variations of the stresses from one side of the cube to the other are of higher order in the cube distances dx_i, the only way a torque balance can be achieved is if $\sigma_{32} = \sigma_{23}$.

Some housekeeping

- So far, we've derived 4 equations (mass conservation and momentum conservation $\times 3$)
- We have 10 unknowns, however (3 components of velocity, density, and 6 independent components of the stress tensor)
- We clearly do not have enough to close our dynamical description

- We can make more progress by injecting more information about the stress tensor. In particular, the constitutive relations for the stress tensor that describe the relation between the stress tensor and the state of the fluid flow.
- Up until now, the equations we have derived apply equally well to steel as to any fluid.
- It is high time for some description of the stress tensor that reflects the basic definition of a fluid and the properties of fluids like water and air that are of principal interest to us...

3.2 Putting the stress tensor in diagonal form

A key step in formulating the equations of motion **for a fluid** requires specifying the stress tensor in terms of the properties of the flow (and in particular the velocity field) so that the theory becomes closed. That is, the number of variables is reduced to the number of governing equations.

A key step towards this aim is to recognize that because the stress tensor is symmetric, we can always find a coordinate frame in which the stresses are purely normal (where the stress tensor is diagonal).

Aside: In practice, finding this frame amounts to diagonalizing the stress tensor σ_{ij} (which in the geographical $x-y-z$ frame is generally not diagonal to begin with). This is done by finding a transformation matrix a_{ij} which renders σ_{ij} diagonal in a new frame. This exercise yields an eigenvalue problem for the σ_j in the diagonalized tensor and defines three eigenvalues corresponding to the three diagonal elements of the new stress tensor. For each eigenvalue, there will be an eigenvector. Sine the stress tensor is symmetric, the eigenvectors corresponding to different eigenvalues are orthogonal and define the coordinate frame in which σ is diagonal (purely normal stresses).

We won't review this diagonalization exercise here, although it is a standard topic in most introductory linear algenbra courses if you want a refresher.

Instead, we will move on secure in the knowledge that because the stress tensor is symmetric ($\sigma_{ij} = \sigma_{ji}$) we can always find a frame in which the stress tensor is diagonal,

$$\sigma_{ij} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \tag{3.1}$$

3.3 A bit of an aside... The static (or hydrostatic) pressure and the deviatoric stress

Recall that one definition of a fluid is a material that, if subject to forces or stresses that will not lead to a change of volume, deforms at a constant rate and so will not remain at rest.

In follows that in a fluid at rest, the stress tensor must have only diagonal terms. Further, the stress tensor would have to be diagonal in any coordinate frame because the fluid doesn't know which frame we choose to use to describe the stress tensor.

The only 2^{nd} order stress tensor that is diagonal in **all** frames is one in which each diagonal element is the same. We define that value as the static pressure P (sometimes called the hydrostatic pressure – this is the pressure in a fluid at rest). In this case, the stress tensor is:

$$\sigma_{ij} = -P\delta_{ij} = -P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.2}$$

Where δ_{ij} is the Kronecker delta ($\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$).

The fact that the stress tensor is isotropic implies that the normal stress in any orientation is always $-P$, and the tangential stress is always zero. (This isotropy of pressure is often called Pascal's Law).

When the fluid is moving, the pressure (often defined as the average normal force on a fluid element) need not be the thermodynamic pressure. That is,

$$\sigma_{jj}/3 = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \tag{3.3}$$

where σ_{jj} is the trace of the stress tensor (think back to Einstein notation!) and the RHS is the definition of pressure in terms of the elements of the stress tensor for a moving fluid.

This definition is useful because the trace of the stress tensor σ_{jj} is invariant under coordinate transformations (As you will show in an assignment!).

The deviatoric stress

We can always split the stress tensor into two parts and write it as:

$$\sigma_{ij} = -P\delta_{ij} + \tau_{ij} \tag{3.4}$$

where τ_{ij} (the deviatoric stress) is simply defined as the difference between the pressure and the total stress tensor. **If** we define the pressure as the average normal stress as above, **then** the trace of the deviatoric stress tensor must be zero (but if the pressure is so defined the we cannot guarantee it is equal to the thermodynamic pressure and we will have to represent the difference of the two also in terms of the fluid motion).

On the other hand, if we define the pressure as the thermodynamic pressure, then the trace of $\tau_{ij} \neq 0$. Of course, it is a matter of choice which of these we choose. The final equations will be the same.

3.4 The analysis of fluid motion at a point

We are next going to try to relate the stress tensor to some property of fluid motion. In almost all cases of interest to us, this relationship will be a local one. (This is an important property true for simple fluids).

Our approach will be to analyze the nature of the flow in the vicinity of an arbitrary point and discover what aspects of the motion will determine the stress.

Consider the fluid motion near the point x_i. Within a small neighbourhood of that point and using the continuous nature of fluid motion we can represent the velocity in terms of a Taylor series. If the neighbourhood under consideration is small, only the first term of this expansion is important.

$$u_i(x_j + \delta x_j) = u_i(x_j) + \frac{\partial u_i}{\partial x_j}\delta x_j \tag{3.5}$$

where $\frac{\partial u_i}{\partial x_j}$ is the deformation tensor and $\frac{\partial u_i}{\partial x_j}\delta x_j = \delta u_i(x_j)$ is the velocity deformation. Note that both the velocity and the displacement δx_i are vectors, so $\frac{\partial u_i}{\partial x_j}$ is a second order tensor.

We can rewrite the velocity deviation as the sum of a symmetric tensor, $\delta u_i^{(\mathrm{s})}$ and an anti-symmetric part, $\delta u_i^{(\mathrm{a})}$.

That is, $\delta u_i = \delta u_i^{(\mathrm{s})} + \delta u_i^{(\mathrm{a})}$, where:

$$\delta u_i^{(\mathrm{s})} = e_{ij}\delta x_j = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\delta x_j \tag{3.6}$$

$$\delta u_i^{(\mathrm{a})} = \xi_{ij}\delta x_j \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\delta x_j \tag{3.7}$$

where e_{ij} is the rate of strain tensor and ξ_{ij} is related to the vorticity.

3.5 The vorticity

Consider the anti-symmetric part of the velocity deviation, $\xi_{ij}\delta x_j$. ξ_{ij} has three components,

$$\begin{aligned}
&\begin{pmatrix} i=2 \\ j=1 \end{pmatrix} \quad \xi_{21} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) = -\xi_{12} \\
&\begin{pmatrix} i=2 \\ j=1 \end{pmatrix} \quad \xi_{32} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) = -\xi_{23} \\
&\begin{pmatrix} i=1 \\ j=3 \end{pmatrix} \quad \xi_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) = -\xi_{31}
\end{aligned} \tag{3.8}$$

These are the three components of the vector $\frac{1}{2}\vec{\omega} = \frac{1}{2}\nabla \times \vec{u}$, where $\omega \equiv \mathrm{curl}(\vec{u})$.

Or, in index notation, $\omega_i = \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}$. We call ε_{ijk} the alternating tensor (Or Levi-Civita). $\varepsilon_{ijk} = 1$ if (i,j,k) is an even permutation of $(1,2,3)$, $\varepsilon_{ijk} = -1$ if (i,j,k) is an odd permutation of $(1,2,3)$, and $\varepsilon_{ijk} = 0$ if any index is repeated.

We call $\vec{\omega} \equiv$ Vorticity. As we shall see, vorticity occupies a central place in the dynamics of atmospheric and oceanic phenomena and it is one of the fundamental portions of the general composition of fluid motion.

The relationship between ξ_{ij} and the vorticity is fairly straightforward. For example, $\xi_{32} = \frac{\omega_1}{2}$, with the other components following cyclically. In general,

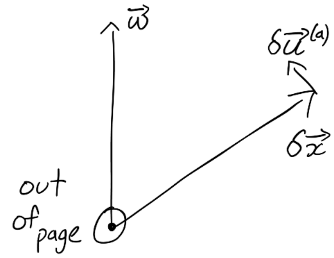
$$\xi_{ij} = -\frac{1}{2}\varepsilon_{ijk}\omega_k \tag{3.9}$$

The anti-symmetric part of the velocity deviation $\delta u_i^{(\mathrm{a})}$ is then

$$\delta u_i^{(\mathrm{a})} = \xi_{ij}\delta x_j = -\frac{1}{2}\varepsilon_{ijk}\omega_k\delta x_j = -\frac{1}{2}\varepsilon_{ikj}\omega_k\delta x_j \tag{3.10}$$

This may be more recognizable written in vector form,

$$\delta\vec{u}^{(\mathrm{a})} = \frac{1}{2}\vec{\omega} \times \delta\vec{x} \tag{3.11}$$

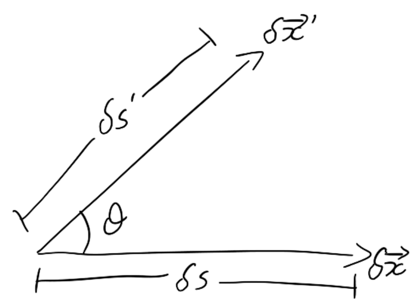


Physically, $\delta\vec{u}^{(a)}$ represents the deviation velocity due to a pure rotation at a rotation rate which is one half the local value of the vorticity. We recognize that it is a pure rotation because the associated velocity vector $\delta\vec{u}^{(a)}$ is always perpendicular to the displacement $\delta\vec{x}$ so there is no increase in the length of $\delta\vec{x}$, only a change in direction.

3.6 The rate of strain tensor

Now consider the contribution of the symmetric part of the velocity deviation, $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$. This is the rate of strain tensor.

To see the connection to the rate of strain, consider two differential line element vectors, $\delta\vec{x}$ and $\delta\vec{x}'$, originating at the same point separated by angle θ. The their respective lengths be δs and $\delta s'$:



Now, assuming the common origin is following the fluid, each displacement vector will change depending on changes in the difference between the position of the origin and the position of the tip of the vector.

$$\frac{\mathrm{D}}{\mathrm{D}t}\delta x_i = \delta u_i \tag{3.12}$$

Now consider the inner product of our two displacement vectors, $\delta x_i \delta x_i' = \delta s \delta s' \cos\theta$. The time rate of change of this inner product is then:

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{D}t}\delta x_i \delta x_i' &= \frac{\mathrm{D}}{\mathrm{D}t}\left(\delta s \delta s' \cos\theta\right) \\ &= \left(\delta s \frac{\mathrm{D}}{\mathrm{D}t}\delta s' + \delta s' \frac{\mathrm{D}}{\mathrm{D}t}\delta s\right)\cos\theta - \sin\theta \frac{\mathrm{D}\theta}{\mathrm{D}t}\delta s \delta s'\end{aligned} \tag{3.13}$$

We can also write $\frac{\mathrm{D}}{\mathrm{D}t}\delta x_i \delta x_i' = \delta x_i \delta u_i' + \delta x_i' \delta u_i$ (by applying product rule). By substituting $\delta u_i' = \frac{\partial u_i}{\partial x_j}\delta x_j'$ and $\delta u_i = \frac{\partial u_i}{\partial x_j}\delta x_j$,

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{D}t}\delta x_i \delta x_i' &= \delta x_i \frac{\partial u_i}{\partial x_j}\delta x_j' + \delta x_i' \frac{\partial u_i}{\partial x_j}\delta x_j \qquad \text{interchanging dummy indices } i \text{ and } j \text{ in the last term} \\ &= \delta x_i \delta x_j' \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \\ &= 2 e_{ij} \delta x_i \delta x_j'\end{aligned} \tag{3.14}$$

Equating the RHS of equation 3.13 and equation 3.14 and dividing by $\delta s \delta s'$ (since we assume neither is singular) yields:

$$\cos\theta\left(\frac{1}{\delta s'}\frac{\mathrm{D}}{\mathrm{D}t}\delta s' + \frac{1}{\delta s}\frac{\mathrm{D}}{\mathrm{D}t}\delta s\right) - \sin\theta\frac{\mathrm{D}\theta}{\mathrm{D}t} = 2e_{ij}\frac{\delta x_i}{\delta s}\frac{\delta x_j'}{\delta s'} \tag{3.15}$$

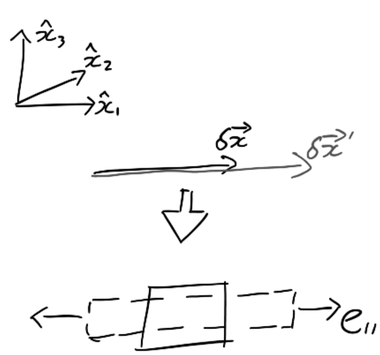
where $\frac{\delta \vec{x}}{\delta \vec{x}'}$ and $\frac{\delta x_j'}{\delta s'}$ are unit vectors in the directions of $\delta\vec{x}$ and $\delta\vec{x}'$. We can use this equation to interpret the components of the tensor e_{ij}.

Example 3.1. Let $\delta\vec{x}$ and $\delta\vec{x}'$ be co-linear so that $\theta = 0$ and let $\delta\vec{x}$ lie along the x_1 axis, $\delta\vec{x} = (\delta x_1, 0, 0)$. If $\theta = 0$, then $\cos\theta = 1$, $\sin\theta = 0$, and $\delta s = \delta x_1$. Equation 3.15 then implies:

$$\frac{1}{\delta x_1'}\frac{\mathrm{D}}{\mathrm{D}t}\delta x_1' + \frac{1}{\delta x_1}\frac{\mathrm{D}}{\mathrm{D}t}\delta x_1 \tag{3.16}$$

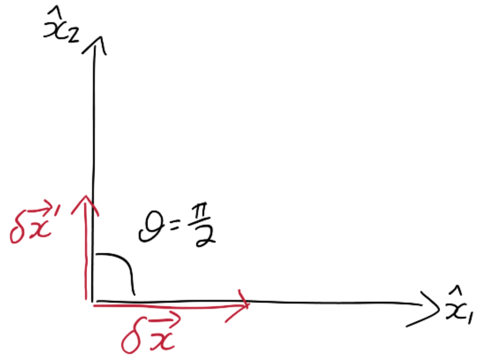
If we take δx_1 and $\delta x_1'$ to be the same length,

$$\frac{1}{\delta x_1}\frac{\mathrm{D}}{\mathrm{D}t}\delta x_1 = e_{11} \tag{3.17}$$



Physically, the diagonal elements of the rate of strain tensor represent the rate of stretching of a fluid element along the corresponding axis.

Example 3.2. Now choose $\delta\vec{x}$ and $\delta\vec{x}'$ to lie along the x_1 and x_2 axes respectively, so $\delta\vec{x} = (\delta x_1, 0, 0)$ and $\delta\vec{x}' = (0, \delta x_2, 0)$.

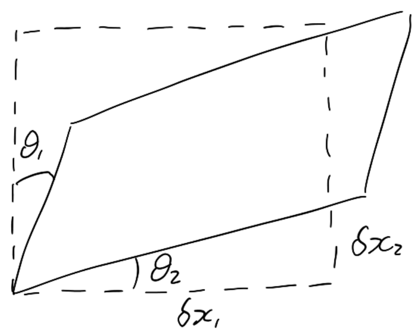


$\theta = \pi/2$ implies that $\cos\theta = 0$ and $\sin\theta = 1$. Equation 3.15 then implies:

$$-\frac{\mathrm{D}\theta}{\mathrm{D}t} = 2e_{12} \qquad \left(\frac{\delta x_i}{\delta s}\frac{\delta x'_j}{\delta s'} \quad \text{is only non-zero when } i=1 \text{ and } j=2\right) \tag{3.18}$$

A little geometry shoes that:

$$-\frac{\mathrm{D}\theta}{\mathrm{D}t} = \frac{\mathrm{D}\theta_1}{\mathrm{D}t} + \frac{\mathrm{D}\theta_2}{\mathrm{D}t} \tag{3.19}$$



Since, for example, $\tan\theta_2 \approx \theta_2 = \frac{\delta x_2}{\delta x_1}$ (for θ_2 small). Thus, $\frac{\mathrm{D}\theta_2}{\mathrm{D}t} = \frac{1}{\delta x_1}\frac{\partial u_2}{\partial x_1}\delta x_1$, where $\frac{\partial u_2}{\partial x_1}\delta x_1 = \delta x_2$.

And so,

$$-\frac{\mathrm{D}\theta}{\mathrm{D}t} = 2e_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \tag{3.20}$$

Physically, we can interpret the off-diagonal element of the rate of strain tensor as the rate or shearing strain of a fluid element.

3.7 Principal Strain Axes and the Decomposition of the Motion

As in the case of the stress tensor, the rate of strain tensor can also be diagonalized. We can always find a frame in which $e_{ij} = e_{(j)}\delta_{ij}$. This is called the principal strain axes frame, or simply the principal axes frame.

In this frame, the velocity deviation associated with the symmetric part of the deformation tensor is a pure strain along the principal axes. Lines parallel to the principal axes are strained (extended or contracted) and not rotated.

$$e_{jj} = \frac{\partial u_j}{\partial x_j} = \nabla \cdot \vec{u} = \text{rate of volume expansion} \tag{3.21}$$

We can thus write the rate of strain tensor in the principal axes frame as the sum of two parts:

$$e_{ij} = \underbrace{e_{(i)}\delta_{ij} - \frac{1}{3}\left(\nabla \cdot \vec{u}\right)\delta_{ij}}_{\substack{\text{Pure strain without} \\ \text{any change in volume}}} + \underbrace{\frac{1}{3}\left(\nabla \cdot \vec{u}\right)\delta_{ij}}_{\text{Pure volume change}} \tag{3.22}$$

Combining our results, [the velocity deviation associated with the anti-symmetric part of the deformation tensor (rotation associated with the vorticity)] and [the velocity deviated associated with the velocity deviation associated with the symmetric part of the deformation tensor (pure strain without any change in volume plus a pure volume change] and [a pure translation (accounting for a translation of the origin of the two displacement vectors)]. We see that we can always represent the motion of a fluid element in terms of three basic parts:

(1) = a pure translation

(2) = a pure strain along the principal axes

(3) = a pure rotation (associated with the vorticity)

This fact was demonstrated by Helmholtz in 1858.

Example 3.3 (Decomposition of the motion for a simple linear shear flow)**.** Consider a simple linear shear flow in the x_1 direction only, for which the velocity has the form $\vec{u} = \left(x_2 \frac{\partial u_1}{\partial x_2}, 0, 0\right)$.

Consider first the motion described by the rate of strain tensor. By definition,

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\begin{pmatrix} 2\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} & 2\frac{\partial u_2}{\partial x_2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix} \tag{3.23}$$

because in this problem, $u_1 = mx_2$, $u_2 = 0$, $\frac{\partial u_1}{\partial x_1} = 0$, $\frac{\partial u_1}{\partial x_2} = m$.

We can find the principal axes and strain rates by diagonalizing e_{ij}. The principal strain rates are given by the eigenvalues $e^{(\mathrm{j})}$ roots of the characteristic equation defined by the condition for the vanishing determinant of the

2×2 matrix,

$$
\begin{aligned}
e_{ij} - e^{(\mathrm{j})} I = 0 \\
\begin{vmatrix} 0 - e^{(\mathrm{j})} & \frac{1}{2}m \\ \frac{1}{2}m & 0 - e^{(\mathrm{j})} \end{vmatrix} = 0 \\
\left(e^{(\mathrm{j})}\right)^2 = \left(\frac{1}{2}m\right)^2 = 0 \\
e^{(\mathrm{j})} = \pm\frac{1}{2}m
\end{aligned}
\tag{3.24}
$$

So $e^{(1)} = +\frac{1}{2}m$ and $e^{(2)} = -\frac{1}{2}m$. The are the principal strain rates (the elements of the rate of strain tensor in diagonal form).

The principal axes are the eigenvectors $(a_{j1}, a_{j2})^{\mathrm{T}}$ that satisfy:

$$
\begin{aligned}
(e_{ij} - e^{(\mathrm{j})}) \begin{pmatrix} a_{j1} \\ a_{j2} \end{pmatrix} = 0 \\
\begin{pmatrix} 0 - e^{(\mathrm{j})} & \frac{1}{2}m \\ \frac{1}{2}m & 0 - e^{(\mathrm{j})} \end{pmatrix} \begin{pmatrix} a_{j1} \\ a_{j2} \end{pmatrix} = 0
\end{aligned}
\tag{3.25}
$$

For $e^{(1)} = +\frac{1}{2}m$:

$$
\begin{aligned}
-\frac{1}{2}ma_{j1} + \frac{1}{2}ma_{j2} = 0 \\
-a_{j1} + a_{j2} = 0
\end{aligned}
\tag{3.26}
$$

$$
\text{Choose} \quad a_{j1} = a_{j2} = \frac{1}{\sqrt{2}} \quad \text{So the vector has unit length}
$$

The first eigenvector is any two element column vector in which the two elements have equal magnitude and sign.

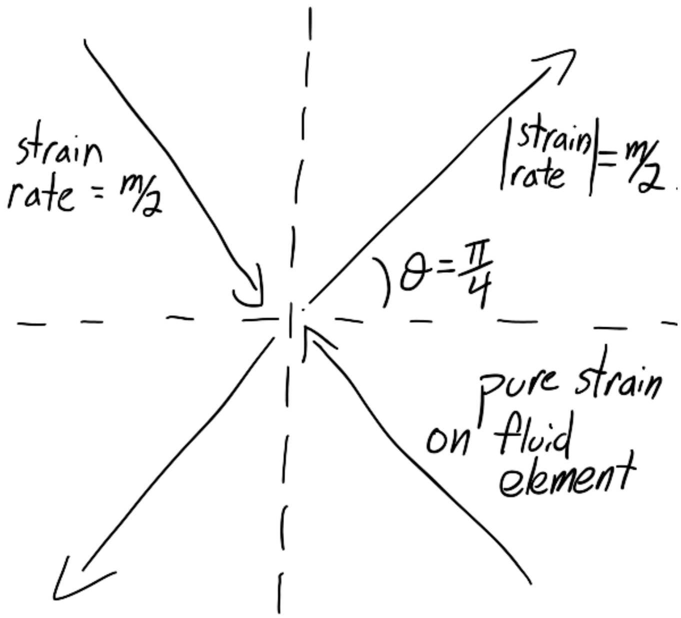
For $e^{(2)} = -\frac{1}{2}m$:

$$
\begin{aligned}
\frac{1}{2}ma_{j1} + \frac{1}{2}ma_{j2} = 0 \\
a_{j1} + a_{j2} = 0 \\
a_{j1} = -a_{j2}
\end{aligned}
\tag{3.27}
$$

$$
\text{Choose} \quad a_{j1} = \frac{1}{\sqrt{2}} \quad \text{and} \quad a_{j2} = -\frac{1}{\sqrt{2}} \quad \text{So the vector has unit length}
$$

The second eigenvector is any two element column vector in which the two element have equal magnitude and opposite sign.

Thus, the principal axes are defined by the orthogonal eigenvectors $\frac{1}{2}(1,1)^{\mathrm{T}}$ and $\frac{1}{2}(1,-1)^{\mathrm{T}}$. The rate of strain along these axis directions is given by the eigenvalues $e^{(1)}$ and $e^{(2)}$, equal to $+\frac{1}{2}m$ and $-\frac{1}{2}m$, respectively.



Next, the rotation of the fluid element related to the vorticity: By definition,

$$\begin{aligned}
\omega_i &= \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j} \qquad \left(\text{Or, if you prefer in vector notation,} \quad \nabla\times\vec{u} = \begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix}\right) \\
&= \begin{pmatrix} \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ -m \end{pmatrix}
\end{aligned} \tag{3.28}$$

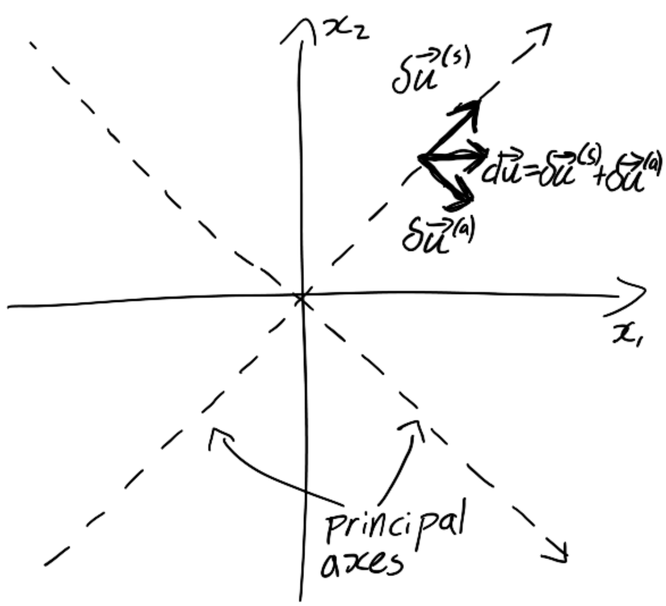
where we have used the fact that $u_1 = mx_2$, $u_2 = u_3 = 0$, $\frac{\partial u_1}{\partial x_1} = \frac{\partial u_1}{\partial x_3} = 0$, and $\frac{\partial u_1}{\partial x_2} = m$.

We can now find the velocity deviations due to the rate of strain and vorticity. We will consider the velocity deviation associated with a small displacement in the direction of the first principal axis.

$$\begin{aligned}\delta u_i^{(\mathrm{s})} &= e_{ij}\delta x_j \\ &= \frac{1}{2}\begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{\sqrt{2}}{4}\begin{pmatrix} m \\ m \end{pmatrix}\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}\delta u_i^{(\mathrm{a})} &= \frac{1}{2}\varepsilon_{ikj}\omega_k\delta x_j \qquad \left(\text{or in vector format,} \quad \frac{1}{2}\vec{\omega}\times d\vec{x}\right) \\ &= \begin{vmatrix} 0 & 0 & -\frac{1}{2}m \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{vmatrix} \\ &= \frac{\sqrt{2}}{4}\begin{pmatrix} m \\ -m \\ 0 \end{pmatrix}\end{aligned} \tag{3.30}$$



As a summary: The principal axes are those along which the fluid element feels *pure strain*. $\delta\vec{u}^{(\mathrm{s})}$ is the velocity deviation due to this pure strain. $\delta\vec{u}^{(\mathrm{a})}$ is the velocity deviation due to rotation. Together, they sum to the net velocity deviation $\vec{u}$.

3.8 The relation between stress and rate of strain

Recall that we are still looking for the relationship between the three independent components of the stress tensor σ_{ij} and the fluid velocity to reduce the number of unknowns in our equations.

Thus far, we have considered the velocity deviation in the immediate vicinity of an arbitrary point and expressed in in terms of the **deformation tensor**, $\frac{\partial u_i}{\partial x_j}$, which we've further decomposed into a symmetric part, the **rate of strain tensor**, $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ and an antisymmetric part, $\xi_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$, with the **vorticity**, $\omega_i = \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}$ defined through $\xi_{ij} = \frac{1}{2}\varepsilon_{ikj}\omega_k$.

To make further progress, we need to make some assumptions about our fluid that seem to be valid based on our experience with "normal" fluids like air and water with which be are most concerned...

These assumptions are:

① The stress tensor is a function only of the deformation tensors $\frac{\partial u_i}{\partial x_j}$ and various thermodynamic state functions like temperature. The stress depends only on the spatial distribution of velocity near the element under consideration and thermodynamics. That is, **the stress-rate of strain relation is local**.

② The fluid is homogeneous in the sense that **the relationship between stress and rate of strain (rate of change of deformation) is the same everywhere**. There is spatial variation in the stress σ_{ij} only due to spatial variations on the deformation tensor $\frac{\partial u_i}{\partial x_j}$. (*This distinguishes a fluid from a solid, for which the stress tensor depends on the strain itself.*)

③ **The fluid is isotropic**. The relation between stress and rate of strain has no preferred spatial orientation. Obviously, given a particular rate of strain, as in the previous example, there will be a special direction for the stress. But this is only because of the geometry of the strain field and **not** because of the structure of the fluid. This is not true for certain fluids with long chain molecules in their structures for which rates of strain along the direction of the chains give stresses different than in other directions.

Aside:

Assumptions ①, ②, and ③ collectively define what is called a Stokesian Fluid (after George Gabriel Stokes). One important property of a Stokesian Fluid is that the principal axes of the stress and the principal axes of strain must coincide.

(4) **The relationship between stress and rate of strain is linear**. This assumption defines a Newtonian fluid. A Newtonian fluid is one in which the viscous stresses arising from its flow, at every point, are linearly proportional to the local rate of strain. Water and air satisfy this assumption. Ketchup and toothpaste don't!

If assumptions (1) through (4) are satisfied, it implies we are searching for a general relation between the deviatoric stress and the deformation tensor of the form:

$$\tau_{ij} = T_{ijkl} \frac{\partial u_k}{\partial x_l} \tag{3.31}$$

where T_{ijkl} is a 4^{th} order proportionality tensor that satisfies the conditions of homogeneity and isotropy. This implies that T_{ijkl} is independent of orientation in space, and any spatial structure to stress must reflect the spatial structure of the deformation.

One can show that this most general 4^{th} order isotropic tensor has the form:

$$T_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{3.32}$$

(*One can also* appreciate the evasion of this derivation... Ask if you want to see the proof, or trust blindly!)

From this, it follows that the stress tensor has the form:

$$\begin{aligned} \sigma_{ij} &= -P\delta_{ij} + \tau_{ij} \\ &= -P\delta_{ij} + (\alpha\delta_{ij}\delta_{kl} + \beta(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})) \frac{\partial u_k}{\partial x_l} \\ &= -P\delta_{ij} + \alpha\delta_{ij}\frac{\partial u_k}{\partial x_k} + \beta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial u_i}\right) \end{aligned} \tag{3.33}$$

The traditional notation uses λ for α and μ for β, so it is common to write the stress tensor as:

$$\boxed{\sigma_{ij} = -P\delta_{ij} + 2\mu e_{ij} + \lambda e_{kk}\delta_{ij}} \tag{3.34}$$

This is the stress tensor defined in terms of pressure and the rate of strain tensor which is valid for a fluid that satisfies a local stress-rate of strain relation ((1)), is homogeneous ((2)), is isotropic ((3)), and has a linear stress-rate of strain relation ((4)).

Some notes to keep in mind:

- The stress tensor depends on P and e_{ij} but not on the vorticity/antisymmetric part of the deformation tensor.
- The 3^{rd} term, $\lambda e_{kk}\delta_{ij}$, is proportional to the divergence of the velocity field ($e_{kk} = \frac{\partial u_k}{\partial x_k}$ when thinking of Einstein notation). Recall that the divergence is related to the rate of volume change. If the fluid is

incompressible (which water is on the scale of ocean motions, for example), then this term is zero. Even if the fluid is compressible (like air) in dynamics of interest, the deviatoric stress is important when the shear is large (usually near boundaries) and typically dominates this relatively small divergence term.

Back to the definition of P: Equation 3.34 says that the stress tensor σ_{ij} depends on three scalars: P, μ, and λ. As we have discussed, there are two obvious definitions of P that need not be the same: The average normal stress and the thermodynamic pressure.

If we define P as the average normal stress (momentarily we will label this as $\bar{P}$), then

$$P = \bar{P} = -\frac{1}{3}\sigma_{ii} = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \tag{3.35}$$

This is a *mechanical definition of pressure*. It implies that the trace of the deviatoric stress must be zero:

$$\begin{aligned} \mathrm{Tr}(\tau_{ij}) = \tau_{ii} &= 2\mu e_{ii} + \lambda e_{kk}\delta_{ii} \\ &= e_{jj}(2\mu + 3\lambda) \quad \text{(by swapping dummy indices)} \\ &= 0 \end{aligned} \tag{3.36}$$

This implies a relation between λ and μ,

$$\lambda = -\frac{2}{3}\mu. \tag{3.37}$$

But, at the end of the day, we will have to write out equations in a way that links the mechanics and thermodynamics, and we will want to introduce the equilibrium thermodynamic pressure P_e.

$\bar{P}$ and P_e will differ by an amount depending on the normal stresses of the deformation tensor that arise from the fluid's motion. More precisely, in an isotropic medium, the difference between the two definitions of pressure is:

$$\bar{P} = P_e - \eta\frac{\partial u_i}{\partial x_j}\delta_{ij} = P_e - \eta\left[dvu_j x_j\right] = P_e - \eta\nabla\cdot\vec{u} \tag{3.38}$$

so the stress tensor in terms of the thermodynamic pressure becomes:

$$\boxed{\sigma_{ij} = P_e\delta_{ij} + 2\mu e_{ij} + (\eta - 2/3\mu)e_{kk}\delta_{ij}} \tag{3.39}$$

Of course, if we defined P as the thermodynamic in equation 3.34 we would have arrived at the result:

$$\sigma_{ij} = P_e\delta_{ij} + 2\mu e_{ij} + \lambda e_{kk}\delta_{ij} \tag{3.40}$$

(So the relation between the two formulations tells us $\lambda = \eta - 2/3\mu$.)

Some notes:

- P and P_e differ when the fluid is composed of complex molecules with internal degrees of freedom. In this case, the stress can also depend on the rate at which the fluid density is changing with time.

- We see this dependence as a contribution to the pressure in the form $\eta(\nabla \cdot \vec{u})$ where $\nabla \cdot \vec{u}$ is a measure of the rate of density change (it represents the rate of change of the fluid volume per unit volume as seen by an observer moving with the fluid).

- For the fluid of interest to us, η can be quite large: approximately 3μ for water and 100μ for air at STP. But the effect on the flow of this term involving η and $\nabla \cdot \vec{u}$ is usually small even in compressible flows. This is true except when density changes over vary small distances (in shock waves for example) or very short timescales (in high intensity ultrasound for example).

- In "practical" GFD, it is common to take $\eta = 0$ and neglect this term,

$$\sigma_{ij} = -P_e \delta_{ij} + 2\mu e_{ij} + \lambda e_{kk} \delta_{ij} \tag{3.41}$$

  where $\lambda = -2/3\mu$.

- This form of the stress tensor was derived first by the fluid dynamicist Claude-Louis Navier in 1822 without the divergence terms. Other derivations followed by Cauchy (1828), Poisson (1829), Saint-Venant (1843), and finally a form more closely resembling the above by Stokes (1845). The resulting fluid momentum equations with the full stress tensor in this form are called the Navier-Stokes Equations.

3.9 The Navier-Stokes Equations

Now that we have an explicit form for the stress tensor in terms of the pressure and the fluid velocity (specifically the spatial gradient of the fluid velocity in the form of the deformation tensors), we can write the momentum equation in a form suitable for a fluid. Using equation 3.41 and substituting into the momentum equation **??**, we get:

$$\rho \frac{Du_i}{Dt} = \rho F_i - \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}(2\mu e_{ij}) + \frac{\partial}{\partial x_i}\left(\lambda \frac{\partial u_k}{\partial x_k}\right) \tag{3.42}$$

or equivalently, by expanding e_{ij} in terms of the velocity gradients:

$$\boxed{\rho \frac{Du_i}{Dt} = \rho F_i - \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right) + \frac{\partial}{\partial x_i}\left(\lambda \frac{\partial u_k}{\partial x_k}\right)} \tag{3.43}$$

These are the full Navier-Stokes equations in index notation! Here (and henceforth) we have dropped the subscript e on the pressure and we are assuming P is the thermodynamic pressure and not the average normal stress. As they stand, it is difficult to write these equations completely in vector form, but with some notational looseness,

$$\rho \frac{D\vec{u}}{Dt} = \rho \vec{F} - \nabla P + \mu \nabla^2 \vec{u} + (\lambda + \mu) \nabla \left(\nabla \cdot \vec{u} \right) + \left(\nabla \lambda \right) \left(\nabla \cdot \vec{u} \right) + \hat{i}_i 2 e_{ij} \frac{\partial \mu}{\partial x_j} \tag{3.44}$$

which, if the fluid is incompressible **and** if the temperature variations in the fluid are small enough so the viscosity can be approximated as a constant, becomes the more familiar:

$$\boxed{\rho \frac{D\vec{u}}{Dt} = -\nabla P + \rho \vec{F} + \mu \nabla^2 \vec{u}} \tag{3.45}$$

Some Housekeeping

- We currently have four equations (mass conservation and three momentum equations) and five unknowns. So we are still one equation short. In simple cases, we can assume density is constant. If this assumption is not appropriate, we need a way to relate density to the pressure field. To do this, we will have to consider coupling the dynamical equations we have derived to continuum statements of the laws of thermodynamics.

- Note that the total time derivative $\frac{D\vec{u}}{t}$ contains the nonlinear $\vec{u} \cdot \nabla \vec{u}$, which is awkward to express in other coordinates (such as spherical coordinates, which is particularly relevant to describing fluid motions on a rotating planet...)

- We use the identity $\vec{u} \cdot \nabla \vec{u} = \vec{\omega} \times \vec{u} + \nabla |\vec{u}|^2/2$ to derive another form of the Navier-Stokes equations:

$$\rho \left(\frac{\partial u}{\partial t} + \vec{\omega} \times \vec{u} \right) = \rho \vec{F} \left(\nabla P + \rho \nabla |\vec{u}|^2/2 \right) + \mu \nabla^2 \vec{u} \tag{3.46}$$

- Once again, note that the Navier-Stokes Equations are nonlinear due to the advection of momentum by the velocity field.

- It can also be helpful to note that if the fluid is incompressible, you can show that the frictional force $\mu \nabla^2 \vec{u}$ can be written as $-\mu \nabla \times \vec{\omega}$. So you can express friction force in terms of the vorticity! It's that neat?