EOSC 579 - Chapter 1 - Stratification
Lecture 1 : Stratification

For much of what you considered in E512, you took the density to be homogeneous. In the real ocean, the density increases with depth. Often, near the surface and bottom there are nearly homogeneous mixed layers and in the interior there are one or more regions of sharp density gradients (pycnoclines). To be mathematically tractable we usually treat the stratification as changing linearly (as we will do today) or as a series of stacked homogeneous layers. Near the end of E512, you developed the Quasi-geostrophic equation for a stratified fluid. Let's review that as a way to bring it back to mind, and a way to introduce my notation. As its a review, I will skip some details (as noted below).

We want to form a vorticity equation starting from the full-stratified equations.

We are thinking about the interior of ocean so we will neglect friction. We are including stratification and we want to further divide the variable density ρ' into two components: $\rho_*(z)$ a background stratification and $\tilde{\rho}(x, y, z, t)$ the perturbations around that density. So $\rho' = \rho_* + \tilde{\rho}$ and $|\rho_*| \gg |\tilde{\rho}|$. Also note that $\partial p_*/\partial z = -\rho_* g$ is independent of (x, y) and so does not contribute to horizontal pressure gradients.

Thus our equations are:

$$\frac{D_h u}{Dt} - fv = \frac{-1}{\rho_o}\frac{\partial p}{\partial x} \qquad (1a)$$

$$\frac{D_h v}{Dt} + fu = \frac{-1}{\rho_o}\frac{\partial p}{\partial y} \qquad (1b)$$

$$\frac{\partial \tilde{p}}{\partial z} = -\tilde{\rho} g \qquad (1c)$$

$$\nabla \cdot \vec{u} = 0 \qquad (1d)$$

$$\frac{\partial \tilde{\rho}}{\partial t} + \vec{u}_h \cdot \nabla \tilde{\rho} + w\frac{\partial \rho_*}{\partial z} + w\frac{\partial \tilde{\rho}}{\partial z} = 0 \qquad (1e)$$

where in (1e) the last term is much smaller than the second last and so can be neglected. Let $N^2 = -g/\rho_o \partial\rho_*/\partial z$ where N is the Brunt-Väisälä frequency. Then (1e) becomes:

$$\frac{D_h \tilde{\rho}}{Dt} = \frac{\rho_o N^2}{g} w \qquad (2)$$

We now form a vorticity equation from (1a) and (1b). To do so we take $\frac{\partial}{\partial y}$ of (1b) and subtract $\frac{\partial}{\partial x}$ of (1a). Group terms we get:

$$\frac{D_h}{Dt}(\zeta + f) + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(\zeta + f) = 0 \tag{3}$$

Consider the scale for $\zeta = \mathcal{U}/\mathcal{L}$ compared to f. $\mathcal{O}(\zeta/f) = Ro$. Assuming $Ro \ll 1$ we can neglect ζ in favour of f in the second term (not the derivatives).

We now consider conservation of volume (1d) to give that

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -\frac{\partial w}{\partial z} \tag{4}$$

Differentiating (2) with respect to z we get:

$$\frac{\partial w}{\partial z} = \frac{g}{\rho_o}\frac{\partial}{\partial z}\frac{1}{N^2}\frac{D_h\tilde{\rho}}{Dt} \tag{5}$$

But D_h/Dt consists of derivatives with respect to x, y, t and not z so reverse order of differentiation noting that N^2 is only a function of z.

$$\frac{\partial w}{\partial z} = \frac{g}{\rho_o}\frac{D_h}{Dt}\left[\frac{\partial}{\partial z}\left(\frac{\tilde{\rho}}{N^2}\right)\right] \tag{6}$$

But from (1c) $\tilde{\rho} = -\partial\tilde{p}/\partial z/g$ so:

$$\frac{\partial w}{\partial z} = -\frac{1}{\rho_o}\frac{D_h}{Dt}\left[\frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial\tilde{p}}{\partial z}\right)\right] \tag{7}$$

Substituting into (3) above we get:

$$\frac{D_h}{Dt}(\zeta + f) + \frac{f}{\rho_o}\frac{D_h}{Dt}\left[\frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial\tilde{p}}{\partial z}\right)\right] = 0 \tag{8}$$

Making a *Quasi-geostrophic* approximation, we can approximate ζ in this equation by

$\zeta_g = \partial v_g/\partial x - \partial u_g/\partial y$. And approximate $D_h/Dt = D_g/Dt$.[1] The geostrophic velocities are:

$$v_g = \frac{1}{\rho_o f_o}\frac{\partial \tilde{p}}{\partial x}, \qquad u_g = -\frac{1}{\rho_o f_o}\frac{\partial \tilde{p}}{\partial y} \tag{9}$$

where f_o is the constant middle value for f and so

$$\zeta_g = \frac{1}{\rho_o f_o}\nabla^2 \tilde{p} \tag{10}$$

and remembering that $D_g f/Dt = \beta v_g$ we get a single equation for the pressure:

$$\frac{D_g}{Dt}\nabla^2\tilde{p} + \frac{D_g}{Dt}\left[\frac{\partial}{\partial z}\left(\frac{f_o^2}{N^2}\frac{\partial \tilde{p}}{\partial z}\right)\right] + \beta\frac{\partial \tilde{p}}{\partial x} = 0 \tag{11}$$

Note that we can write this in the form

$$\frac{\partial q}{\partial t} + \mathcal{J}(\psi, q) = 0 \tag{12}$$

$$q = \nabla^2\psi + \beta y + \frac{\partial}{\partial z}\left(\frac{f_o^2}{N^2}\frac{\partial \psi}{\partial z}\right) \tag{13}$$

where $\psi = \rho_o f \tilde{p}$.

The stratified quasi-geostrophic equation is separable. Let $\tilde{p} = \Pi(z)Q(x,y,t)$ and substitute:

$$\Pi\frac{D_g}{Dt}\nabla^2 Q + \frac{D_g Q}{Dt}\left[\frac{\partial}{\partial z}\left(\frac{f_o^2}{N^2}\frac{\partial \Pi}{\partial z}\right)\right] + \Pi\beta\frac{\partial Q}{\partial x} = 0 \tag{14}$$

Divide through by $\Pi D_g Q/Dt$ and move second term to RHS.

$$\frac{\frac{D_g}{Dt}\nabla^2 Q + \beta\frac{\partial Q}{\partial x}}{\frac{D_g Q}{Dt}} = -\frac{1}{\Pi}\left[\frac{\partial}{\partial z}\left(\frac{f_o^2}{N^2}\frac{\partial \Pi}{\partial z}\right)\right] \tag{15}$$

Now this is hardly a simple equation BUT the LHS is not a function of z and the RHS is only a function of z (not x, y, t). Now if z varies the LHS does not vary and thus the RHS cannot vary either: it must be equal to a constant. Thus both side must equal a constant.

[1]One should do this properly scaling the terms but I refer you to your detailed QG derivation in EOSC 512 or see Cushman-Roisin

This constant has units of one over length squared so set it to $1/a^2$.

So then we have two equations. A horizontal and a vertical equation: linked only by the constant a.

$$\frac{D_g}{Dt}\left(\nabla^2 Q - \frac{Q}{a^2}\right) + \beta\frac{\partial Q}{\partial x} = 0 \tag{16a}$$

$$\frac{\partial}{\partial z}\left(\frac{f_o^2}{N^2}\frac{\partial \Pi}{\partial z}\right) + \frac{\Pi}{a^2} = 0 \tag{16b}$$

Vertical Equation (16b)

In order to solve the vertical equation we will need boundary conditions at the top and bottom of the domain.

Boundary condition at the bottom of the domain : no flow through the bottom $\vec{u_H} \cdot \nabla_H h + w = 0$ where $z = -h(x, y)$ is the bottom.

Boundary condition at the top of the domain : vertical flow makes the surface go up and down $w = D\eta/Dt$.

Upper Boundary Condition

$$w = D\eta/Dt,\ z = \eta \tag{17}$$

We need to linearize this equation. So $D\eta/Dt$ becomes $\partial\eta/\partial t$ and we can evaluate at $z = 0$ the undisturbed surface height, rather than at $z = \eta$.

$$w = \frac{\partial \eta}{\partial t},\ z = 0 \tag{18}$$

Right at the surface the pressure is simply due to the deflection of the surface. So $p = \rho_o g \eta$ and thus

$$w = \frac{1}{\rho_o g}\frac{\partial p}{\partial t},\ z = 0 \tag{19}$$

and remember from (2) $w\rho_o N^2/g = -D_h\tilde{\rho}/Dt$ which linearized is $w = g/(\rho_o N^2)\partial\tilde{\rho}/\partial t$. Using the hydrostatic equation gives

$$w = -\frac{1}{\rho_o N^2}\frac{\partial^2 p}{dz dt} \tag{20}$$

Combining and writing $p = Q\Pi$ gives

$$N^2 \frac{\partial Q}{\partial t} \Pi = -g \frac{\partial Q}{\partial t} \frac{\partial \Pi}{\partial z}, \; z = 0 \tag{21}$$

or

$$N^2 \Pi = -g \frac{\partial \Pi}{\partial z}, \; z = 0 \tag{22}$$

Lower Boundary Condition

$$\vec{u_h} \cdot \nabla_h h + w = 0, \; z = -h \tag{23}$$

We will use the quasi-geostrophic equations because it is easier (but it can be done for full linear swe). Substitute for u and v using the geostrophic assumption and for w from (20):

$$-\frac{1}{\rho_o f} \frac{\partial p}{\partial y} \frac{\partial h}{\partial x} + \frac{1}{\rho_o} f \frac{\partial p}{\partial x} \frac{\partial h}{\partial y} - \frac{1}{\rho_o N^2} \frac{\partial^2 p}{dz dt}, \; z = -h \tag{24}$$

Write $p = Q\Pi$

$$N^2 \Pi \left(\frac{\partial Q}{\partial y} \frac{\partial h}{\partial x} - \frac{\partial Q}{\partial x} \frac{\partial h}{\partial y} \right) + f \frac{\partial \Pi}{\partial z} \frac{\partial Q}{\partial t} = 0, \; z = -h \tag{25}$$

This equation poses a problem. Q does not simply factor out. We have a mixture of Q and Π and a mixed evaluation (at h which varies with x and y). The only way out of this dilemma is to assume $h = h_o$ a constant $\rightarrow$ the bottom is flat. Then

$$f \frac{\partial \Pi}{\partial z} \frac{\partial Q}{\partial t} = 0, \; z = -h_o \tag{26}$$

or

$$\frac{\partial \Pi}{\partial z} = 0, \; z = -h_o \tag{27}$$

The vertical and horizontal separation for the Quasi-geostrophic and the Shallow Water Equations only works if the fluid has constant depth.