4.2 EOSC 579 - Chapter 4 - Lecture 2 : Stommel Circulation

4.2.1 Learning Goals

At the end of this lecture you will be able to:

- derive the Stommel solution by separation of variables

4.2.2 The Problem

Assuming a homogeneous flat-bottom ocean, linear, steady state and quasi-geostrophic gives the Stommel vorticity equation

$$\beta v = \frac{1}{\rho H} \hat{k} \cdot \nabla \times \vec{\tau} - \frac{f\delta}{2H}\zeta \tag{1}$$

Assume a wind $\tau_1 = -\tau_o \cos(\pi y/b)$, $\tau_2 = 0$ and a basin $x \in [0, a]$ and $y \in [0, b]$.

As the flow is quasi-geostrophic we can introduce a streamfunction

$$v = \frac{\partial \psi}{\partial x} \tag{2a}$$

$$u = -\frac{\partial \psi}{\partial y} \tag{2b}$$

So

$$\beta \frac{\partial \psi}{\partial x} + \frac{f\delta}{2H}\nabla^2 \psi = -\frac{\tau_o \pi}{\rho H b} \sin\left(\frac{\pi y}{b}\right) \tag{3}$$

with boundary conditions, ψ constant on the boundaries. Take the constant as 0. Rewrite the equation as

$$\nabla^2 \psi + B \frac{\partial \psi}{\partial x} = \gamma \sin\left(\frac{\pi y}{b}\right) \tag{4}$$

where $B = 2\beta H/f\delta$ and $\gamma = -2\pi\tau_o/\rho\delta f b$.

4.2.3 Solution

The general solution is the sum of a particular solution and the solution of the homogeneous equation. Try $\psi_p = Q \sin\left(\frac{\pi y}{b}\right)$ as a particular solution. Substitute:

$$-\left(\frac{\pi}{b}\right)^2 Q \sin\left(\frac{\pi y}{b}\right) = \gamma \sin\left(\frac{\pi y}{b}\right) \tag{5}$$

Therefore $Q = -\gamma(b/\pi)^2$.

The homogeneous equation is

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + B\frac{\partial \psi}{\partial x} = 0 \tag{6}$$

Use separation of variables and let $\psi = X(x)Y(y)$. Substituting

$$X''Y + XY'' + BX'Y = 0 \tag{7}$$

Rearranging

$$-\frac{1}{X}\left(X'' + BX'\right) = \frac{Y''}{Y} = \text{const} \equiv -\alpha^2 \tag{8}$$

Taking the Y equation:

$$Y'' + \alpha^2 Y = 0 \tag{9}$$

gives $Y = C\cos\alpha y + D\sin\alpha y$. We have boundary conditions:

$\psi = 0$ at $y = 0$ which implies $Y = 0$ at $y = 0$, *ie.*, $C = 0$.

and $\psi = 0$ at $y = b$ which implies $Y = 0$ at $y = b$, *ie.*, $D\sin\alpha b = 0$, $\alpha b = n\pi$.

$$\alpha = \frac{n\pi}{b}, n = 1, 2, ... \tag{10}$$

Taking the X equation:

$$X'' + BX' - \left(\frac{n\pi}{b}\right)^2 X = 0 \tag{11}$$

this is an ODE with constant coefficients, Let $X = \exp \lambda x$. Substitute:

$$\lambda^2 + B\lambda - \left(\frac{n\pi}{b}\right)^2 = 0 \tag{12}$$

There are two solutions for each n and using the quadratic formula,

$$\lambda_n = -\frac{B}{2} + \left[\left(\frac{n\pi}{b}\right)^2 + \frac{B^2}{4}\right]^{1/2} > 0 \tag{13a}$$

$$\mu_n = -\frac{B}{2} - \left[\left(\frac{n\pi}{b}\right)^2 + \frac{B^2}{4}\right]^{1/2} < 0 \tag{13b}$$

For each $n, n = 1, 2...$ we have a solution

$$X_n = C_n \exp(\lambda_n x) + D_n \exp(\mu_n x). \tag{14}$$

and so the general solution to the homogeneous equation is

$$\psi = \sum_{n=1}^{\infty} \left(C_n \exp(\lambda_n x) + D_n \exp(\mu_n x)\right) \sin\left(\frac{n\pi y}{b}\right) \tag{15}$$

where, without loss of generality we have set $D = 1$.

Total solution is

$$\psi = \sum_{n=1}^{\infty} \left(C_n \exp(\lambda_n x) + D_n \exp(\mu_n x)\right) \sin\left(\frac{n\pi y}{b}\right) - \gamma \left(\frac{b}{\pi}\right)^2 \sin\left(\frac{\pi y}{b}\right) \tag{16}$$

We have satisfied the boundary conditions at the north and south coasts but need to satisfy the boundary conditions at $x = 0, a$, *ie.*, $\psi = 0$. At $x = 0$:

$$0 = \sum_{n=1}^{\infty} (C_n + D_n) \sin\left(\frac{n\pi y}{b}\right) - \gamma \left(\frac{b}{\pi}\right)^2 \sin\left(\frac{\pi y}{b}\right) \text{ for all } y, 0 \leq y \leq b \tag{17}$$

Which implies, for $n = 1$:

$$C_1 + D_1 = \gamma \left(\frac{b}{\pi}\right)^2 \tag{18}$$

and for $n > 1$:

$$C_n + D_n = 0 \tag{19}$$

At $x = a$:

$$0 = \sum_{n=1}^{\infty} (C_n \exp(\lambda_n a) + D_n \exp(\mu_n a)) \sin\left(\frac{n\pi y}{b}\right) - \gamma \left(\frac{b}{\pi}\right)^2 \sin\left(\frac{\pi y}{b}\right) \tag{20}$$

which implies, for $n = 1$:

$$C_1 \exp(\lambda_1 a) + D_1 \exp(\mu_1 a) = \gamma \left(\frac{b}{\pi}\right)^2 \tag{21}$$

and for $n > 1$:

$$C_n \exp(\lambda_n a) + D_n \exp(\mu_n a) = 0 \tag{22}$$

Consider the two equations for $n > 1$, substitute first into second:

$$C_n (\exp(\lambda_n a) - \exp(\mu_n a)) = 0 \tag{23}$$

But $\mu_n \neq \lambda_n$ so $C_n = D_n = 0$!

We can solve for C_1 and D_1 for the two equations above: $\exp(\lambda_1 a)$ * first - second :

$$D_1 (\exp(\lambda_1 a) - \exp(\mu_1 a)) = \gamma \left(\frac{b}{\pi}\right)^2 (\exp(\lambda_1 a) - 1) \tag{24}$$

and $\exp(\mu_1 a)$ * first - second :

$$C_1 \left(-\exp(\lambda_1 a) + \exp(\mu_1 a)\right) = \gamma \left(\frac{b}{\pi}\right)^2 \left(\exp(\mu_1 a) - 1\right) \tag{25}$$

The final solution is:

$$\psi = \gamma \left(\frac{b}{\pi}\right)^2 \sin\left(\frac{\pi y}{b}\right) \left(P \exp(\lambda_1 x) + Q \exp(\mu_1 x) - 1\right) \tag{26}$$

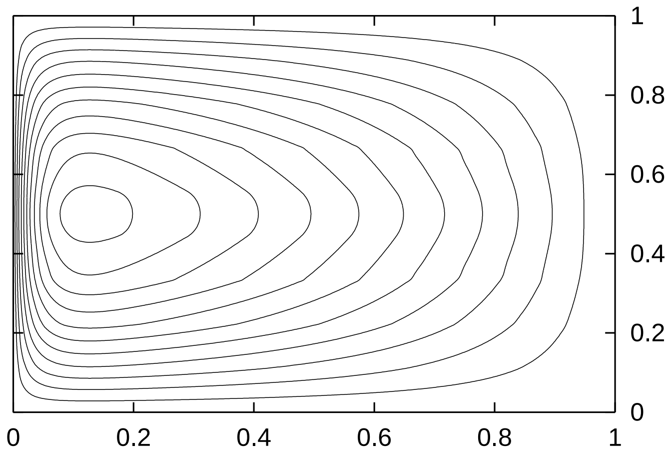
where

$$P = \frac{\exp(\mu_1 a) - 1}{\exp(\mu_1 a) - \exp(\lambda_1 a)} \tag{27}$$

$$Q = \frac{\exp(\lambda_1 a) - 1}{\exp(\lambda_1 a) - \exp(\mu_1 a)} \tag{28}$$

$$\lambda_1 = -\frac{B}{2} + \left[\left(\frac{\pi}{b}\right)^2 + \frac{B^2}{4}\right]^{1/2} > 0 \tag{29}$$

$$\mu_1 = -\frac{B}{2} - \left[\left(\frac{\pi}{b}\right)^2 + \frac{B^2}{4}\right]^{1/2} < 0 \tag{30}$$



Stommel solution for fairly wide western boundary current.