EOSC 579 - Chapter 1 - Stratification
Lecture 2 : Normal Modes

So we have:

$$\frac{\partial}{\partial z}\left(\frac{f^2}{N^2}\frac{\partial \Pi}{\partial z}\right) + \frac{\Pi}{a^2} = 0 \tag{1}$$

where

$$N^2\Pi = -g\frac{\partial \Pi}{\partial z},\ z = 0 \tag{2}$$

$$\frac{\partial \Pi}{\partial z} = 0,\ z = -h_o \tag{3}$$

Special Case, N constant

Assume N is constant. Then (1) is a simple second order ordinary differential equation with constant coefficients. Solutions are of the form $\exp(i\alpha z)$. Substitute:

$$-\alpha^2\frac{f^2}{N^2} + \frac{1}{a^2} = 0 \tag{4}$$

So $\alpha = \pm N/fa$. We can write complete solutions as

$$A\sin(\alpha z) + B\cos(\alpha z) \tag{5}$$

and assume α is positive without lose of generality. Substitute in (2) to give

$$N^2\left(A\sin(0) + B\cos(0)\right) = -g\alpha\left(A\alpha\cos(0) - B\alpha\sin(0)\right) \tag{6}$$

or $N^2B = -g\alpha A$. Substitute in (3) gives

$$A\alpha\cos[\alpha(-h_o)] - B\alpha\sin[\alpha(-h_o)] = 0 \tag{7}$$

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Substitute from first condition into the second to give

$$A\alpha\cos[\alpha(-h_o)] + g\frac{\alpha^2}{N^2}A\sin[\alpha(-h_o)] = 0 \tag{8}$$

or

$$\frac{N^2}{g\alpha} = \tan[\alpha h_o] \tag{9}$$

Typical deep ocean $N = 0.004$ s^{-1}, typical $h_o = 4000$ km and $g = 10$ m s^{-1}. So plot $1.6 \times 10^{-6}/x$ versus $\tan(4000x)$:

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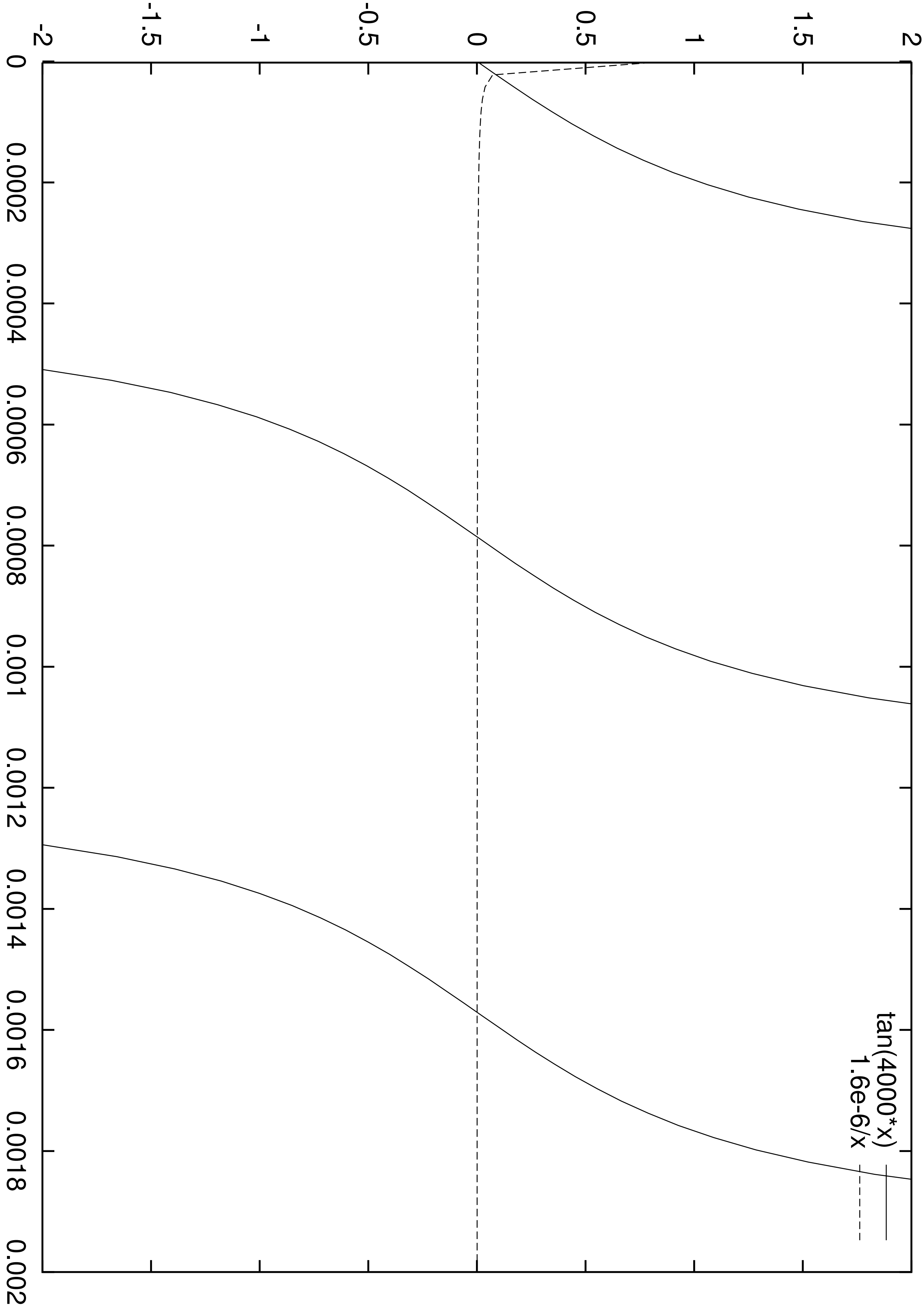

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Notes:

1. There are a number of roots
2. Because N^2/gh_o is small, the first root occurs at very small x where $\tan(h_o x) \approx \alpha x$
3. For the same reason the other roots occur where $N^2/gx \approx 0$.

Barotropic Root

If $\tan(\alpha h_o) \approx \alpha h_o$ then

$$\frac{N^2}{g\alpha} = \alpha h_o \tag{10}$$

and

$$\alpha^2 = \frac{N^2}{gh_o} \tag{11}$$

and as $\alpha = N/fa$

$$a^2 = \frac{gh_o}{f^2} = R^2\,!!! \tag{12}$$

And note that in this case $\alpha h_o = N(h_o/g)^{1/2} \approx 0.08$ and that $A/B = -N^2/(g\alpha) = N(h_o/g)^{1/2} \approx 0.08$ are both small so

$$A\sin(\alpha z) + B\cos(\alpha z) \approx B\cos(0) = B \tag{13}$$

So there is almost no variation in the vertical!

This mode is external mode based upon the pressure due to surface height deflections and in which the whole vertical column of fluid moves together. It has the same speed $(gh_o)^{1/2} \approx 200$ m s^{-1} as a homogeneous fluid and the same lengthscale $(gh_o)^{1/2}/f \approx 2000$ km.

Baroclinic Roots

If $N^2/gx \approx 0$ then

$$\tan(\alpha h_o) = 0 \tag{14}$$

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and

$$\alpha h_o = n\pi,\ n = 1, 2 \ldots \tag{15}$$

and as $\alpha = N/fa$

$$a = \frac{Nh_o}{fn\pi},\ n = 1, 2 \ldots \tag{16}$$

and note that $A/B = -N^2/(g\alpha) = N^2(h_o/gn\pi) \approx 0.002$ for $n = 1$ and smaller for larger n's. So the mode shape

$$A\sin(\alpha z) + B\cos(\alpha z) \approx B\cos(n\pi z/h_o) \tag{17}$$

So for these baroclinic modes, the 'wave speed' af is $Nh_o/n\pi \approx 5/n$ m s^{-1} and the intrinsic lengthscale a is $Nh_o/fn\pi \approx 50/n$ km. The lengthscale Nh_o/f is usually called the baroclinic Rossby radius.

These mode shapes (plus the barotropic mode) look like:

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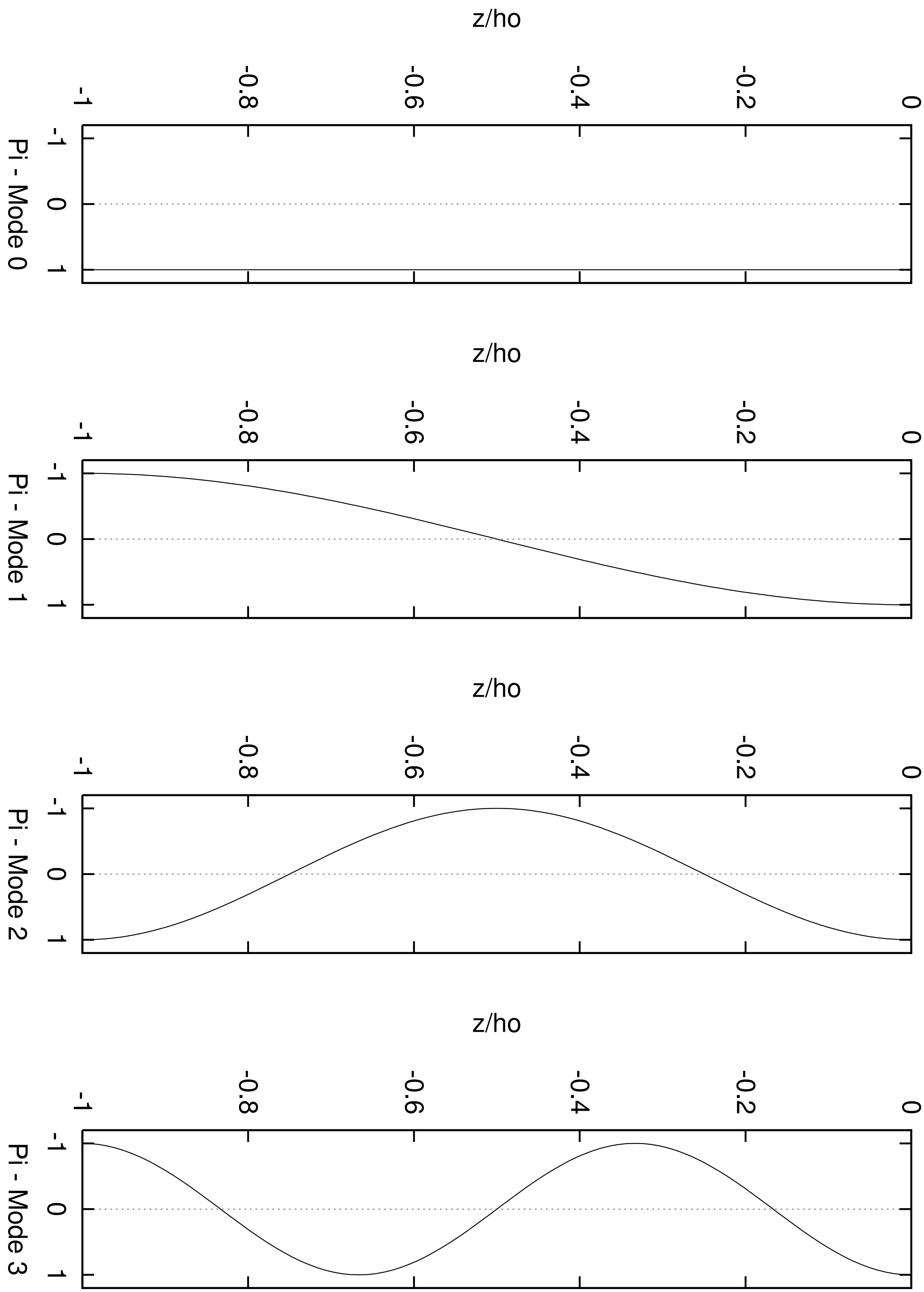

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General Case, N not constant

In the ocean and atmosphere N is not constant. In the ocean stratification is concentrated near the surface. In the case of a non-constant stratification, (1) can be integrated numerically with (2) and (3) as boundary conditions.

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