EOSC 579 - Chapter 3 - Instability
Lecture 3 : Necessary Condition for Instability

Our full quasi-geostrophic equation (Chapter 3, Lecture 2 (6)) with $U = \bar{u} = -\partial\bar{\psi}/\partial y$ is

$$\left[\frac{\partial}{\partial t} + U(y,z)\frac{\partial}{\partial x}\right]\left[\nabla_H^2\psi' + \frac{\partial}{\partial z}\left(\frac{f_o^2}{N^2}\frac{\partial\psi'}{\partial z}\right)\right] + \frac{\partial\psi'}{\partial x}\frac{\partial q}{\partial y} = 0 \tag{1}$$

where

$$\frac{\partial q}{\partial y} = \beta - \frac{\partial^2 U(y,z)}{\partial y^2} - \frac{\partial}{\partial z}\left(\frac{f_o^2}{N^2}\frac{\partial U(y,z)}{\partial z}\right) \tag{2}$$

We assume a perturbation that is wavelike in the x direction but has some unknown shape in y and z. So

$$\psi' = \psi(y,z)\exp[-ik(x-ct)]$$

and substituting gives:

$$(-ikc + ikU)\left[-k^2\psi + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{f^2}{N^2}\frac{\partial\psi}{\partial z}\right)\right] + ik\psi\frac{\partial q}{\partial y} = 0$$

and dividing through by ik gives:

$$(U-c)\left[-k^2\psi + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{f^2}{N^2}\frac{\partial\psi}{\partial z}\right)\right] + \psi\frac{\partial q}{\partial y} = 0 \tag{3}$$

Note that, for a general case, it is not possible to solve (3). A couple of examples of solutions are the Eady problem and the Charney problem.

1.1 Boundary Conditions

We will consider a zonal channel of depth $z = -H$ and north-south width of L. All walls are free-slip (no flow through the wall, but rapid flow possible parallel to the wall).

North-South Boundaries At $y = 0, L$, $v = \partial\psi/\partial x = 0$. Thus ψ is a constant along the wall. We take with constant as zero.

Top and Bottom Boundaries At $z = 0, -H$, $w = 0$. Now from (Chapter 3, Lecture 2, (10)),

$$\begin{aligned}
w &= -\left[\frac{1}{\rho_o N^2}\frac{D_h}{Dt}\left(\frac{\partial p}{\partial z}\right)\right] && (4)\\
&= \frac{-f}{N^2}\frac{D_h}{Dt}\left(\frac{\partial \psi}{\partial z}\right) && (5)\\
&= \frac{-f}{N^2}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\left(\frac{\partial \psi}{\partial z}\right) && (6)\\
&= \frac{-f}{N^2}\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{\partial \psi'}{\partial z}\right) - \frac{\partial \psi'}{\partial x}\frac{\partial U}{\partial z}\right] && (7)\\
&= \frac{-ikf}{N^2}\left[(U-c)\frac{\partial \psi}{\partial z} - \psi\frac{\partial U}{\partial z}\right] && (8)
\end{aligned}$$

So at $z = 0, -H$,

$$(U-c)\frac{\partial \psi}{\partial z} - \psi\frac{\partial U}{\partial z} = 0 \qquad (9)$$

1.2 Finding the Necessary Condition

Assume that the differential equation for ψ (3) and its boundary conditions are unstable and thus support growing wave modes. In this case, $c = c_r + ic_i$ where $c_i \neq 0$ and $c_i > 0$ implies growth.

As U is real $U - c \neq 0$. So consider

$$\int_0^L dy \int_{-H}^0 dz\, \frac{1}{U-c}\, \psi^* \times \text{diff. eqn}$$

where ψ^* is the complex conjugate of ψ.

That gives:

$$\int_0^L dy \int_{-H}^0 dz \left[\psi^* \frac{\partial^2 \psi}{\partial y^2} - k^2|\psi|^2 + f_o^2 \psi^* \frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial \psi}{\partial z}\right) + \frac{|\psi|^2}{U-c}\frac{\partial q}{\partial y}\right] = 0$$

Expand the first term:

$$\int_0^L dy\, \psi^* \frac{\partial^2 \psi}{\partial y^2} = \psi^* \left.\frac{\partial \psi}{\partial y}\right|_0^L - \int_0^L dy \left|\frac{\partial \psi}{\partial y}\right|^2$$

Along $y = 0, L$; ψ must be a constant. Take $\psi = 0$, so

$$\int_0^L dy\, \psi^* \frac{\partial^2 \psi}{\partial y^2} = - \int_0^L dy \left| \frac{\partial \psi}{\partial y} \right|^2$$

At $z = 0, -H$;

$$(U - c)\, \frac{\partial \psi}{\partial z} - \psi\, \frac{\partial U}{\partial z} = 0,$$

So

$$\begin{aligned} \int_{-H}^0 dz\, \psi^* \frac{\partial}{\partial z} \left(\frac{1}{N^2} \frac{\partial \psi}{\partial z} \right) &= \psi^* \frac{1}{N^2} \frac{\partial \psi}{\partial z} \bigg|_{-H}^0 - \int_{-H}^0 dz \frac{1}{N^2} \left| \frac{\partial \psi}{\partial z} \right|^2 \\ &= \frac{|\psi|^2}{N^2 (U - c)} \frac{\partial U}{\partial z} \bigg|_{-H}^0 - \int_{-H}^0 dz \frac{1}{N^2} \left| \frac{\partial \psi}{\partial z} \right|^2 \end{aligned}$$

Substitute these two expressions into the original integral

$$\int_0^L dy \int_{-H}^0 dz \left[- \left| \frac{\partial \psi}{\partial y} \right|^2 - k^2 \left| \psi \right|^2 - \frac{f_o^2}{N^2} \left| \frac{\partial \psi}{\partial z} \right|^2 \right]$$

$$= - \int_0^L dy \int_{-H}^0 \partial z\, \frac{|\psi|^2}{U - c} \frac{\partial q}{\partial y} - f_o^2 \int_0^L dy\, \frac{|\psi|^2}{N^2 (U - c)} \frac{\partial U}{\partial z} \bigg|_{z=-H}^0$$

Now substitute real and imaginary expansion for c, multiply equation by

$$\frac{U - c_r + i c_i}{U - c_r + i c_i}$$

and separate into real and imaginary parts, *ie.,* $A + iB = 0$, which implies both A and B must be zero. B is

$$c_i \left[\int_0^L dy \int_{-H}^0 dz\, \frac{|\psi|^2}{|U - c|^2} \frac{\partial q}{\partial y} + f_o^2 \int_0^L dy\, \frac{|\psi|^2}{N^2 |U - c|^2} \frac{\partial U}{\partial z} \bigg|_{z=-H}^0 \right]$$

For instability $c_i \neq 0$, therefore a necessary, but not sufficient, condition for instability is that [] is zero.

1.3 Barotropic Example

Assume that the flow is independent of depth, so that $\partial U/\partial z = 0$. Then

$$\frac{\partial q}{\partial y} = \beta - \frac{\partial^2 U}{\partial y^2}$$

and our necessary condition is that:

$$\int_0^L dy \, \frac{|\psi|^2}{|U-c|^2} \left(\beta - \frac{\partial^2 U}{\partial y^2} \right) = 0 \tag{10}$$

Now the fraction is necessarily non-negative. So for the integral to be zero, the term in the bracket must be positive in some region of the domain and negative in some other region of the domain.

Lets consider our four examples.

Example 1, 2 $u = 0$ or $u = const$. For either of these the bracket is β which is always positive and so, as we found, these flows are STABLE.

Example 3 $u = Uy/L$ and $\beta = 0$. For this example, the bracket is zero, so the integral is zero and this flow is POTENTIALLY UNSTABLE. We found it was actually stable.

Example 4 $\beta = 0$ but $\partial^2 U/\partial y^2$ is positive (and infinite) at the first change in $\partial U/\partial y$ and negative (and infinite) at the second change in $\partial U/\partial y$. Thus, the bracket changes sign and this flow is POTENTIALLY UNSTABLE. As with most flows for which the bracket is zero, we found this flow is actually unstable.

Note that adding β is often a stabilizing influence; not shown very well here. Example 3 is already stable and as $\partial^2 U/\partial y^2$ goes infinite, no amount of β can stabilize this flow.