EOSC 579 - Chapter 2 - Internal Waves
Lecture 1 : Introduction and Dispersion Relation

1.1 Learning Goals

As the end of this lecture you will be able to:

- describe the internal wave "climate" of the world oceans
- define the term "Garrett Munk" spectrum
- explain which terms should be kept for a simple set of internal wave equations
- derive the dispersion relation for internal waves

This lecture is based on a set of notes written by R. Pawlowicz and all san serif text is almost directly quoted

1.2 Description of Internal Wave "Climate"

As soon as people began making observations of temperature and salinity in the ocean, they discovered that large amplitude fluctuations were omnipresent.

In the 1940's and 1950's, time series began to show that these fluctuations spanned a large range of frequencies from f to N and that even at the lowest frequencies there was very little coherence between even closely spaced sites.

In the 1970's the idea of a *Spectrum* was applied, and an empirical form for this spectrum was developed by Garrett and Munk which seemed to describe the ocean (to within a factor of two or so).

Implications

1. Internal waves are somehow *saturating* the ocean
2. Dissipation much be *weak*
3. Interesting places are those that *differ* from the Garrett-Munk spectrum, because they might indicate generation and/or dissipation. There are four significant regions.
   (a) Arctic Ocean (very low)

(b) near bumps and coastlines

(c) high shear regions

(d) near the equator

1.3 Derivation, Linear Internal Waves

Go back to the full equations, assume Boussinesq but do *not* assume hydrostative. Assume linear flows (that is, neglect advection terms) and assume inviscid.

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_o}\frac{\partial p}{\partial x} \tag{1a}$$

$$\frac{\partial u}{\partial t} + fu = -\frac{1}{\rho_o}\frac{\partial p}{\partial y} \tag{1b}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_o}\frac{\partial p}{\partial z} - \rho' g \tag{1c}$$

$$\frac{\partial \rho'}{dt} + w\frac{\partial \rho_*}{\partial z} = 0 \tag{1d}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1e}$$

Assume each of the five variable act as a wave:

$$u = u_o \exp[i(kx + \ell y + mz - \omega t)]$$

and substitute. Note R_o is the wave amplitude in density.

$$-i\omega u_o - fv_o = -\frac{ik}{\rho_o}p_o \tag{2a}$$

$$-i\omega v_o + fu_o = -\frac{i\ell}{\rho_o}p_o \tag{2b}$$

$$-i\omega w_o = -\frac{im}{\rho_o}p_o - R_o g \tag{2c}$$

$$-i\omega R_o - w_o \rho_o N^2 = 0 \tag{2d}$$

$$iku_o + i\ell v_o + imw_o = 0 \tag{2e}$$

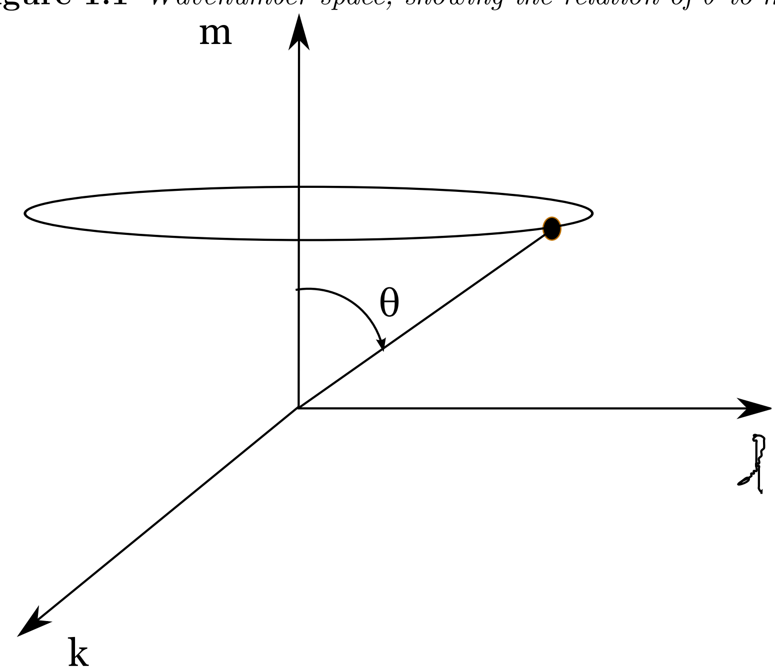
Which, if we assume N is constant, is a set of five linear, homogeneous, algebraic equations in five unknowns. This system only has a non-trivial solution if the "matrix" has a zero determinant. Which after much arthimetic gives:

$$\frac{N^2 - \omega^2}{\omega^2 - f^2} = \frac{m^2}{k^2 + \ell^2}$$

This is a very peculiar dispersion relation. Consider a wave with wave number $\vec{k} = (K, L, M)$. If I reduce the wavelength of this wave by a factor of 2, by increasing all the wavenumbers by a factor of 2 $(2K, 2L, 2M)$, the frequency does not change. Frequency is not a function of the wavelength, but only of the wave direction!

Now m is the vertical wavenumber so the left-hand side is measure of the angle of the wave with the vertical (Figure 1.1).

Figure 1.1 *Wavenumber space, showing the relation of θ to m and k, ℓ.*



So

$$\tan\theta = \frac{(k^2 + \ell^2)^{1/2}}{m}$$

And in terms of this angle we can write

$$\omega^2 = N^2 \sin^2\theta + f^2 \cos^2\theta$$

Which clearly illustrates two things:

1. If we know the stratification, Coriolis parameter and frequency, then the direction of phase propagation to the vertical is given

2. Both the Coriolis frequency and the stratification play a role. The the Coriolis frequency is more important as the waves propagate more vertically and the stratification is more important as the waves propagate more horizontally.

1.4 More Properties from the Dispersion Relation

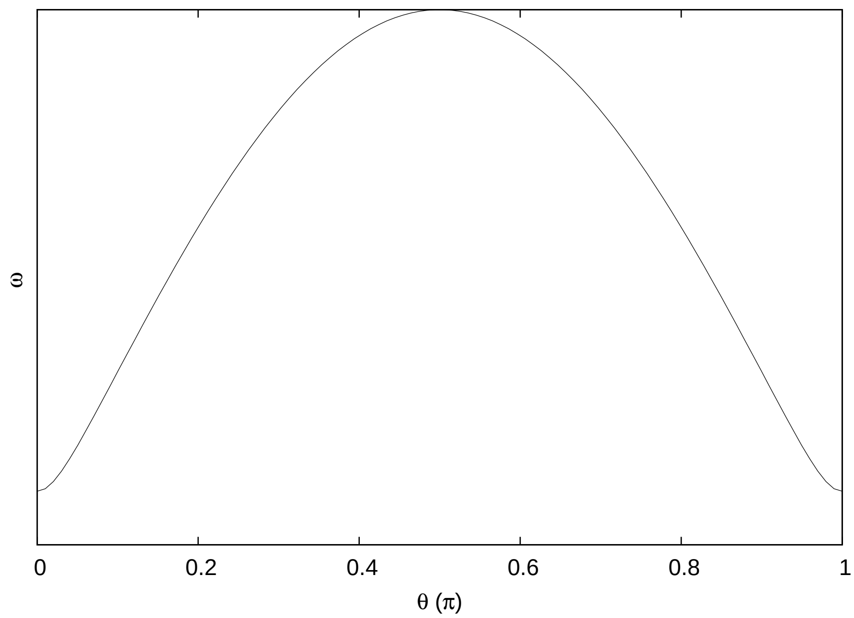
1. Consider (2e):

$$ku + \ell v + mw = 0$$

which means $\vec{k} \cdot \vec{u} = 0$ or the flow is perpendicular to the wave-vector!

2. Group velocity, the direction of energy propagation, is the gradient of omega in $\vec{k}$ space. Looking back at Figure 1.1 we can see that the gradient is either in the direction of increasing or decreasing θ. Note that this gradient is perpendicular to $\vec{k}$ and therefore perpendicular to the phase speed. If we plot ω as function of θ we get Figure 1.2.

Figure 1.2 *Wave frequency as a function of wave angle from the vertical.*



So the maximum frequency is when the $\theta = \pi/2$ or when the wavenumber is horizontal. So for upward wavenumbers $m > 0$, greater frequencies are toward the horizontal which is downward and for downward wavenumber $m < 0$, greater frequencies are toward the horizontal which is upward. (Phase speed up, group speed down and vice-versa).

3. Note also from Figure 1.2 that the group speed goes to zero as the wavenumber approaches either the horizontal or the vertical.

4. Watch them go (Plate 1.1).

Figure 1.1: *Internal waves made in a stratified non-rotating tank by oscillating a stick. By Barry Ruddick and Dave Hebert at Dalhousie* `http://www.phys.ocean.dal.ca/programs/doubdiff/pics/iw1.mpeg` *for more details see* `http://www.phys.ocean.dal.ca/programs/doubdiff/demos/IW1-Lowfrequency.html` *Matlab movie of a packet of internal waves by R. Pawlowicz. On Canvas Site*