# EOSC 579 - Chapter 3 - Instability
# Lecture 1 : Barotropic Instability

Return to our vorticity equation (eqn 3 in Lecture 1):

$$\frac{D_h}{Dt}(\zeta + f) + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(\zeta + f) = 0 \tag{1}$$

We want now to consider a homogeneous, frictionless, flat bottom ocean, so no Ekman flux, no stretching. Thus the second term is zero. Expanding the first term gives us:

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)(\zeta + f) = 0 \tag{2}$$

Consider a steady, mean flow in the x-direction, with no variation in the x-direction,and with zero y-direction mean flow. This mean flow satisfies the governing equation. Is it stable?

Lets consider a perturbation flow $(u', v')$ with vorticity $\zeta' = \partial v'/\partial x - \partial u'/\partial y$. Our total flow and vorticity is thus:

$$u = \bar{u}(y) + u' \tag{3}$$

$$v = v' \tag{4}$$

$$\zeta = \bar{\zeta} + \zeta' \tag{5}$$

$$= -\frac{\partial \bar{u}}{\partial y} + \zeta' \tag{6}$$

Substitute into our equation:

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\left(f - \frac{\partial \bar{u}}{\partial y}\right) \tag{7}$$

$$+ \quad \left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\zeta' \tag{8}$$

$$+ \quad \left(u'\frac{\partial}{\partial x} + v'\frac{\partial}{\partial y}\right)\left(f - \frac{\partial \bar{u}}{\partial y}\right) \tag{9}$$

$$+ \quad \left(u'\frac{\partial}{\partial x} + v'\frac{\partial}{\partial y}\right)\zeta' = 0 \tag{10}$$

where we have split the equation up into 4 parts:

1. (Time rate of change and advection by the mean flow) of (planetary vorticity and mean flow vorticity)
2. (Time rate of change and advection by the mean flow) of (perturbation vorticity)
3. (Advection by the perturbation flow) of (planetary vorticity and mean flow vorticity)
4. (Advection by the perturbation flow) of (perturbation vorticity)

The first term is zero because both the planetary vorticity ($f$) and the mean flow are steady and independent of $x$. This statement is equivalent to stating that the mean flow satisfies the governing equation.

We will make the weakly nonlinear approximation and assume that the perturbation flow is small compared to the mean flow. Then we can neglect the fourth term and we are left with:

$$\underbrace{\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)}_{A}\overbrace{\zeta'}^{B} + v'\underbrace{\frac{\partial}{\partial y}}_{C}\overbrace{\left(\beta y - \frac{\partial \bar{u}}{\partial y}\right)}^{D} = 0 \tag{11}$$

where $A$ is time-dependence and advection by the mean flow, $B$ is the perturbation vorticity, $C$ is advection by the perturbation and $D$ is the total vorticity of the mean flow.

In addition, because the fluid is inviscid and homogeneous, we have

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0 \tag{12}$$

which means we can use a streamfunction with

$$\frac{\partial \psi}{\partial x} = v' \quad (13)$$

$$-\frac{\partial \psi}{\partial y} = u' \quad (14)$$

and we note that $\nabla^2\psi' = \zeta'$. So our equation becomes:

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\nabla^2\psi + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\left(\beta y - \frac{\partial \bar{u}}{\partial y}\right) = 0 \quad (15)$$

where I have dropped the primes.

## 1 Example One, No Mean Flow

Assuming $\bar{u} = 0$ then

$$\frac{\partial}{\partial t}\nabla^2\psi + \beta\frac{\partial \psi}{\partial x} = 0 \quad (16)$$

which has a solution $\psi = \psi_o \exp\left[i\left(kx + \ell y - \omega t\right)\right]$ with

$$\omega = \frac{-\beta k}{k^2 + \ell^2} \quad (17)$$

That is, the solution is Rossby waves under a rigid-lid.

## 2 Example Two, Uniform Mean Flow

Assuming $\bar{u}$ = a constant then[1]

$$\frac{\partial}{\partial t}\nabla^2\psi + \bar{u}\frac{\partial}{\partial x}\nabla^2\psi + \beta\frac{\partial \psi}{\partial x} = 0 \quad (18)$$

which has a solution $\psi = \psi_o \exp\left[i\left(kx + \ell y - \omega t\right)\right]$ with

$$\omega - k\bar{u} = \frac{-\beta k}{k^2 + \ell^2} \quad (19)$$

[1] last term updated

$$c = \frac{\omega}{k} = \frac{-\beta k}{k^2 + \ell^2} + \bar{u} \tag{20}$$

Now we have a Rossby wave simply advected by the mean flow.

### 2.1 Example Three, Uniform Shear Flow

Assuming $\bar{u} = Uy/L$ over a domain from $y = -L$ to $y = L$ then

$$\left(\frac{\partial}{\partial t} + \frac{Uy}{L}\frac{\partial}{\partial x}\right)\nabla^2\psi + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\left(\beta y - \frac{U}{L}\right) = 0 \tag{21}$$

TOO HARD. Assume an f-plane, that is $\beta = 0$. then

$$\left(\frac{\partial}{\partial t} + \frac{Uy}{L}\frac{\partial}{\partial x}\right)\nabla^2\psi = 0 \tag{22}$$

which most likely means $\nabla^2\psi = 0$. That equation has a solution

$$\psi = \exp\left[i\left(kx - \omega t\right)\right]\left[A\exp(-ky) + B\exp(ky)\right]$$

where I have chosen the solution to be wavelike in $x$ and exponential in $y$ because the domain is unlimited in $x$ but confined in $y$. This form is a solution to equation for any choice of $\omega$ and $k$. Now lets consider the boundary conditions.

At the two walls, the flow must not be through the wall. Thus

$$\frac{\partial \psi}{\partial x} = 0\, at\, \pm L\, \forall\, x, t \tag{23}$$

which implies

$$A\exp(-kL) + B\exp(kL) = 0 \tag{24}$$

$$A\exp(kL) + B\exp(-kL) = 0 \tag{25}$$

The only solution is $A = B = 0$, so there is no wave and no instability.

### 2.2 Example Four, Complicated Enough to be Unstable

Assume an f-plane (as $\beta$-plane math is too hard), an infinite domain, and

$$\bar{u} = -U, \qquad y < -L \tag{26}$$

$$\bar{u} = \frac{Uy}{L}, \quad -L < y < L \tag{27}$$

$$\bar{u} = U, \qquad y > L \tag{28}$$

We will have $\nabla^2\psi = 0$ for all three sections. So for each section we have

$$\psi = \exp\left[i\left(kx - \omega t\right)\right]\left[A_* \exp(-ky) + B_* \exp(ky)\right] \tag{29}$$

**Boundary Conditions**

**1 and 2: Finite Solution** .

- $y \to \infty$, $\psi$ finite, $B_1 = 0$
- $y \to -\infty$, $\psi$ finite, $A_3 = 0$

**3 and 4: v-continuous** .

- $y = L$, v-continuous, $A_1 \exp(-kL) = A_2 \exp(-kL) + B_2 \exp(kL)$
- $y = -L$, v-continuous, $A_2 \exp(kL) + B_2 \exp(-kL) = B_3 \exp(-kL)$

**5 and 6: p-continuous** .

p-continuous at $\pm L$ which implies

$$\frac{\partial p'_\ell}{\partial x}$$

is continuous. Which implies

$$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + v'\frac{\partial \bar{u}}{\partial y} - fv'$$

is continuous. Substituting

$$\left(\bar{u} - \frac{\omega}{k}\right)\frac{\partial \psi}{\partial y} - \frac{\partial \bar{u}}{\partial y}\psi$$

continuous.

Write $\omega/k$ as $c$.

- $y = L$, p=continuous,

$$(\bar{u} - c)A_1(-k)\exp(-kL) =$$

$$(\bar{u} - c)\left[A_2(-k)\exp(-kL) + B_2 k\exp(kL)\right] - \frac{\bar{u}}{L}\left[A_2\exp(-kL) + B_2\exp(kL)\right]$$

- $y = -L$, similarly

The boundary conditions give us four linear homogenous equations in four unknowns: $A_1, A_2, B_2, B_3$. We could write them as a big matrix:

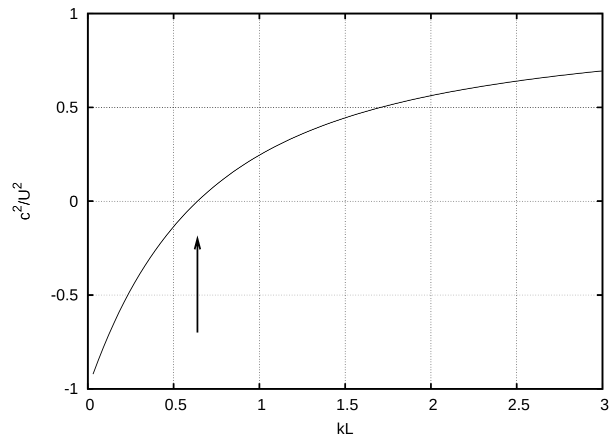
$$\begin{bmatrix} \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{30}$$

where I have not written out the entries in the matrix. The only way to have a non-trivial solution is for the determinant to be zero. That gives:

$$\frac{c^2}{U^2} = \frac{(1 - 2kL)^2 - \exp(-4kL)}{(2kL)^2} \tag{31}$$

which we can plot (Figure 1.1).

**Figure 1.1** *Solution for the phase speed squared for a weakly nonlinear barotropic instability on a limited region shear zone. SEA*




So for $(1-2kL)^2 > \exp(-4kL)$, $c$ is real and if $(1-2kL)^2 < \exp(-4kL)$, $c$ is imaginary. This gives $c$ imaginary for $kL < 0.639$.

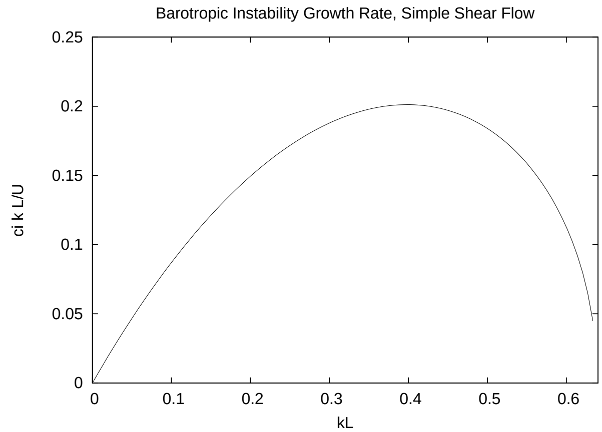
Write $c = c_r + ic_i$, if $c_i$ is greater than zero (which is true if $c$ is imaginary as one of the two roots is positive imaginary and one is negative imaginary), then the solution grows exponentially as

$$\psi = \exp\left[i\left(kx - kct\right)\right]\left[A_* \exp(-ky) + B_* \exp(ky)\right] \quad (32)$$

$$= \exp\left[i\left(kx\right)\right]\exp\left(kc_i t\right)\left[A_* \exp(-ky) + B_* \exp(ky)\right] \quad (33)$$

So the growth rate is $kc_i$ and we can plot that (Figure 1.2).

**Figure 1.2** *Solution for the phase speed squared for a weakly nonlinear barotropic instability on a limited region shear zone. SEA*




Which shows us that the maximum growth rate is for a wave with wavenumber $k = 0.398L$ and this is the wave would expect to see as it would outgrow all the others and dominate.