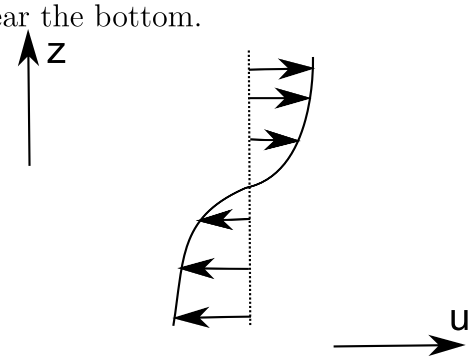
2 Baroclinic Instability and the Eady Problem

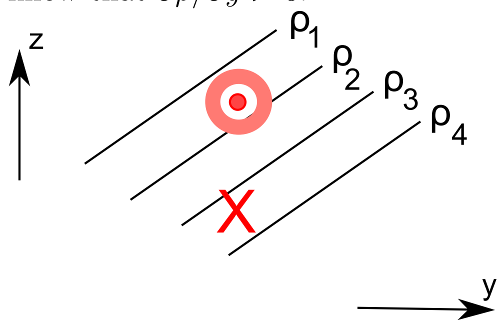
Consider a steady mean flow $U(z)$ where, say, U is positive near the top and U is negative near the bottom.



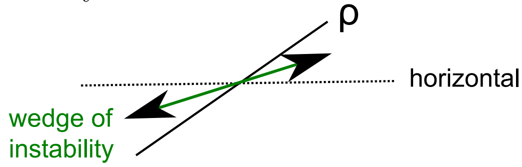
Now by the thermal wind equation:

$$\frac{\partial u}{\partial z} = \frac{g}{\rho_o f}\frac{\partial \rho}{\partial y} \tag{1}$$

we know that $\partial \rho / \partial y > 0$.



There is a lot of potential energy stored here, but geostrophic dynamics can't release it. Pure vertical motion is not unstable, but consider slant motion, inside the wedge of instability as illustrated below.



This figure is a vertical slice.

How do we do this?

Lets start with the stratified, inviscid, quasi-geostrophic equations. (Note that this is first time we've done stratified dynamics in EOSC 579).

$$\left[\frac{\partial}{\partial t} + u_g \cdot \nabla_H\right]\left[f + \zeta_g + \frac{\partial}{\partial z}\left(\frac{f_o}{\rho_o N^2}\frac{\partial p}{\partial z}\right)\right] = 0 \tag{2}$$

Introduce the streamfunction ψ so that

$$u = -\frac{\partial \psi}{\partial y}; v = \frac{\partial \psi}{\partial x} \tag{3a}$$

$$p = \rho_o f_o \psi; \zeta = \nabla_H^2 \psi \tag{3b}$$

Look for a small wave-like feature. So, our total flow = mean flow + small wave. So formally:

$$\psi = \bar{\psi} + \psi', \text{ where } |\psi'| \ll |\bar{\psi}| \tag{4}$$

Substitute and at first only keep the largest terms (no primes):

$$\left[\frac{\partial}{\partial t} - \frac{\partial \bar{\psi}}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \bar{\psi}}{\partial x}\frac{\partial}{\partial y}\right]\left[\beta y + \nabla_H^2 \bar{\psi} + \frac{\partial}{\partial z}\left(\frac{f_o^2}{N^2}\frac{\partial \bar{\psi}}{\partial z}\right)\right] = 0 \tag{5}$$

Given that $\bar{\psi}$ is steady and $\bar{v}$ is zero so $\partial\bar{\psi}/\partial x = 0$, the left-hand bracket gives zero and the equation is solved.

Next terms, those that have one prime in them.

$$\left[\frac{\partial}{\partial t} - \frac{\partial \bar{\psi}}{\partial y}\frac{\partial}{\partial x}\right]\left[\nabla_H^2 \psi' + \frac{\partial}{\partial z}\left(\frac{f_o^2}{N^2}\frac{\partial \psi'}{\partial z}\right)\right] + \frac{\partial \psi'}{\partial x}\frac{\partial}{\partial y}\left[\beta y + \frac{\partial^2 \bar{\psi}}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{f_o^2}{N^2}\frac{\partial \bar{\psi}}{\partial z}\right)\right] = 0 \tag{6}$$

Eady Problem

Assume a mean-shear flow $\bar{U} = Uz/H$, β is negligible, N is constant, and a rigid-lid.

$$\left[\frac{\partial}{\partial t} + U\frac{z}{H}\frac{\partial}{\partial x}\right]\left[\nabla^2 \psi' + \frac{f_o^2}{N^2}\frac{\partial^2 \psi'}{\partial z^2}\right] = 0 \tag{7}$$

Assume a wave-form $\psi' = a(z)\exp[i(\ell x + my - \omega t)]$.

Substitution gives:

$$\left[-i\omega + i\ell U\frac{z}{H}\right]\left[-\ell^2 - m^2 + \frac{f_o^2}{N^2}\frac{\partial^2}{\partial z^2}\right]a = 0 \tag{8}$$

If we assume the first bracket is not zero, we get a simple second-order ODE with constant coefficients; so we get exponentials. Given the sign of ℓ^2 and m^2, they are real exponentials.

Boundary Conditions

The boundary conditions at the top and bottom are no flow through the bottom or top of the ocean: $w = 0$. Given that one of these requirements is at the bottom of the domain ($z = 0$), instead of exponentials we use sinh and cosh. That is:

$$a(z) = A\cosh(\alpha z) + B\sinh(\alpha z) \tag{9}$$

But, our solution is in ψ and our boundary condition is in w. Going back to our original QG derivation, we find:

$$w = \frac{-1}{\rho_o N^2}\frac{D}{Dt}\frac{\partial p}{\partial z} \tag{10}$$

$$= \frac{-f}{N^2}\left[\left(\frac{\partial}{\partial t} + \frac{Uz}{H}\frac{\partial}{\partial x}\right)\frac{\partial \psi'}{\partial z} + \frac{\partial \psi'}{\partial x}\frac{U}{H}\right] \tag{11}$$

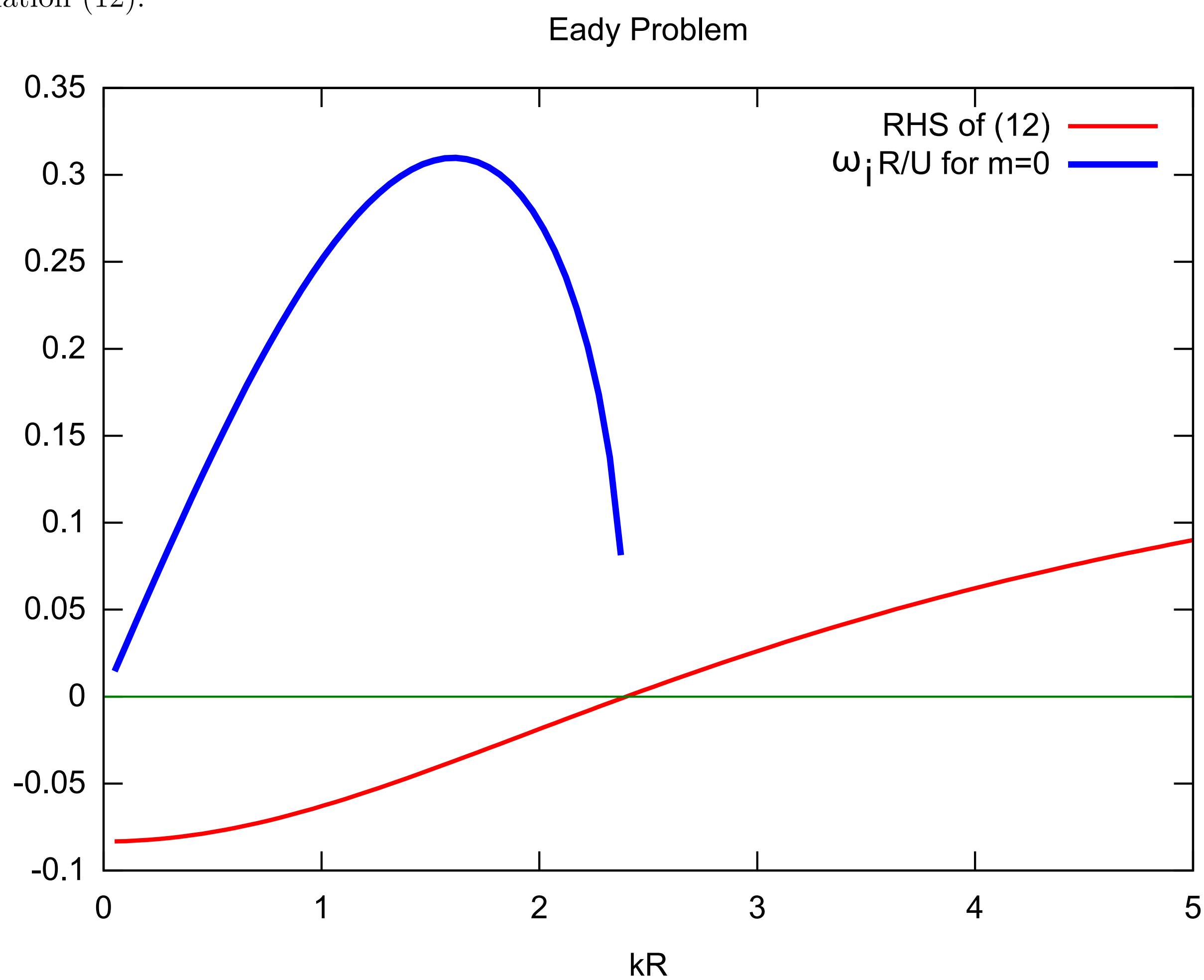
Substituting our solution into the boundary conditions at $z = 0$ and $z = H$ gives two equations for A and B. In order for a non-trivial solution, the determinant must be zero which gives:

$$\left[\frac{\omega}{\ell U} - \frac{1}{2}\right]^2 = \frac{1}{4} + \frac{1}{k^2R^2} - \frac{\coth(kR)}{kR} \tag{12}$$

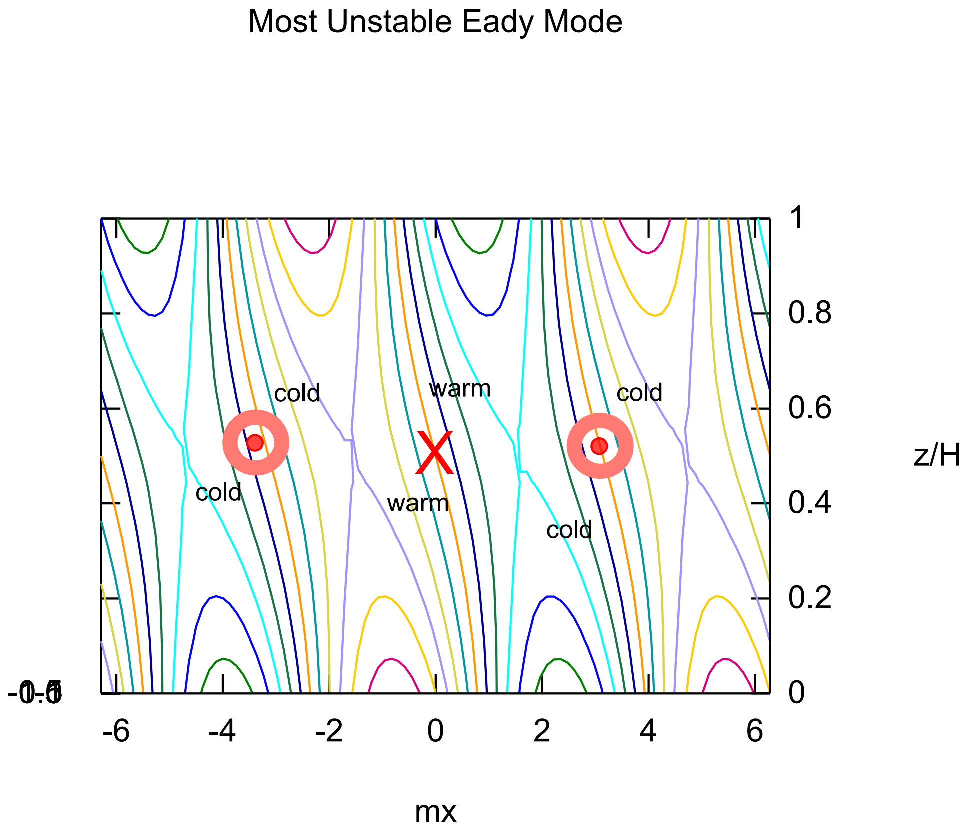
where $k^2 = \ell^2 + m^2$, $\alpha = kN/f$ and $R = NH/f$. Relating B to A using one of the boundary condtions we get the full vertical shape:

$$a(z) = A\left[\cosh\left(\frac{kRz}{H}\right) - \frac{\ell U}{\omega kR}\sinh\left(\frac{kRz}{H}\right)\right] \tag{13}$$

We can investigate the growth characteristics of the wave by looking at the dispersion relation (12):



and the overall structure by looking at the whole wave (at $t = y = 0$) $a(z)exp(imx)$


Most Unstable Eady Mode
cold
warm
cold
cold
warm
cold
X
1
0.8
0.6
0.4
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0
z/H
-6
-4
-2
0
2
4
6
mx